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**SECȚIA I MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1973

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 MECANICĂ TEORETICĂ
 FIZICĂ**

1973

1989

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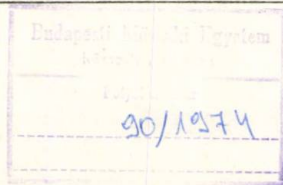
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Tomul XIX (XXIII), fasc. 1—2

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EXTENSIUNEA UNEIA DIN REGULILE LUI HOSPITAL

DE

AL. CLIMESCU

1. Fără a o particulariza notabil, una din regulile lui Hospital se enunță după cum urmează :

I. În condițiile următoare :

a) f definită pe $[0, 1)$, continuă și ≥ 0 ;

b) g idem, dar $g(x) > 0$;

c) $\lim_{x \rightarrow 1} g(x) = \infty$;

d) $\lim_{x \rightarrow 1} \int_0^x g(x) dx = \infty$;

e) $\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = k \geq 0$

are loc egalitatea

$$\lim_{x \rightarrow 1} \frac{\int_0^x f(x) dx}{\int_0^x g(x) dx} = k.$$

Pentru o funcție $g(x)$ dată notăm prin $C(g, k)$ mulțimea funcțiilor $f(x)$ care satisfac la condițiile a) și e).

În demonstrație se utilizează esențial continuitatea celor două funcții, cu toate că e) poate fi verificată pentru funcții $f(x)$ mult mai generale,

chiar neintegrabile. Prin urmare, nu se poate spera să se obțină o generalizare a rezultatului fără să se înlocuiască corespondența

$$f(x) \rightarrow \int_0^x f(x) dx$$

printr-o alta, mai suplă, $f(x) \rightarrow F(x)$, $F(x)$ putînd fi chiar o funcție multivocă.

2. Prin $\varphi(x)$, $\varphi_1(x), \dots, \varphi_n(x)$ notăm funcții caracteristice definite pe $[0, 1]$; $\varphi_1, \varphi_2, \dots, \varphi_n$ vor forma un sistem ortogonal (finit) dacă $\varphi_i(x) \varphi_j(x) = 0$ ($i \neq j$) și $\sum \varphi_i(x) = 1$; evident $\varphi_i^2 = \varphi_i$.

II. $C(g, 0)$ formează un spațiu vectorial real, care se poate extinde la spațiul sumelor $\alpha(x)f(x) + \sum \alpha_i(x)f_i(x)$, notat $S(g, 0)$, unde $f, f_1, \dots, f_n \in C(g, 0)$, $\alpha(x)$ o funcție caracteristică oarecare. Se arată imediat.

3. Următoarele definiții ne vor permite să ajungem la extensiunea propoziției I:

(a) Dacă $f(x) \in C(g, k)$, numim integrala $If(x)$ funcția multivocă

$$\int_0^x f(x) dx + \int_0^x s(x) dx,$$

unde

$$s(x) \in C(g, 0)$$

și

$$\int_0^x s(x) dx \stackrel{\text{def}}{=} \alpha \int_0^x f(x) dx + \sum \alpha_i(x) \int_0^x f_i(x) dx.$$

$If(x)$ este multivocă deoarece $s(x)$ poate fi aleasă arbitrar; pentru $f_1(x), f_2(x), \dots, f_n(x)$ fixate, se va vorbi de o ramură a integralei If .

(b) Numim pseudo-integrală $PIf(x)$ funcția multivocă

$$\beta(x)f(x) + \int_0^x s(x) dx,$$

$\beta(x)$ fiind o funcție caracteristică arbitrar aleasă, dar fixă (în particular se poate lua $\beta(x) = 1$).

Ca funcții multivoce, If și PIf se bucură de proprietățile

$$I(f+g) = If + Ig, \quad PI(f+g) = PIf + PIg,$$

$$I(cf) = cIf, \quad PI(cf) = cPIf$$

și alte proprietăți ale integralei obișnuite se extind cu ușurință.

Are loc propoziția :

III. În condițiile din enunțul I

$$\lim_{x \rightarrow 1} \frac{If(x)}{\int_0^x g(x) dx} = k,$$

oricare ar fi ramura aleasă pentru If .

Dacă condițiile b) — e) sînt verificate, $\{x \mid \beta(x) = 1\}$ este peste tot densă în vecinătatea lui 1 și

$$\frac{\int_0^x g(x) dx}{g(x)} < M,$$

atunci există o infinitate de șiruri $x_n \rightarrow 1$ cu

$$\lim \frac{PIf(x)}{g(x)} = k.$$

Se poate lua ca șir x_n oricare verificînd condiția $\beta(x_n) = 1$.

4. Fie $g(x) > 0$ și strict crescătoare și

$$\int_y^x g(x) dx = (x - y) g[c(x, y)],$$

unde $c(x, y)$ este unic determinat. Definim

$$F(x) = xf[c(x, 0)], \quad F_1(x, y) = (x - y)f[c(x, y)], \quad 0 \leq y < x$$

(evident $F(x) = F_1(x, 0)$). Atunci

IV. În condițiile b) — e), $f(x)$ oarecare, g strict crescătoare au loc relațiile :

$$\lim_{x_n \rightarrow 1} \frac{F(x_n)}{x_n} = k, \quad \lim_{x_{n+1}} \frac{F(x_n, x_{n+1})}{x_{n+1}} = k,$$

$$\int_0^{x_n} g(x) dx \quad \int_{x_n}^{x_{n+1}} g(x) dx$$

$$\left(\frac{1}{x} \int_0^x g(x) dx \right)' = -\frac{1}{x^2} \int_0^x g(x) dx + \frac{g(x)}{x} = \frac{g(x) - g[c(x, 0)]}{x} > 0,$$

deci cînd x_n crește, $c(x_n, 0)$ crește și $\lim c(x_n, 0) = 1$, altfel c) n-ar mai fi verificată.

A doua egalitate rezultă, mai simplu, din

$$\frac{F(x_n, x_{n+1})}{x_{n+1}} = \frac{f[c(x_{n+1}, x_n)]}{g[c(x_{n+1}, x_n)]} \quad \text{și} \quad x_n \leq c(x_{n+1}, x_n) \leq x_{n+1}.$$

$$\int_{x_n} g(x) dx$$

Ca exemplu să luăm $g(x) = \frac{1}{(1-x)^2}$ de unde

$c(x, y) = 1 - \sqrt{(1-x)(1-y)}$, $c(x, 0) = 1 - \sqrt{1-x}$, $F(x) = xf(1 - \sqrt{1-x})$,
de unde, în condițiile obișnuite

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = k \Rightarrow \lim_{x \rightarrow 1} \frac{xf(1 - \sqrt{1-x})}{\frac{1}{1-x}} = k.$$

5. Într-un manuscris rămas de la O. Mayer [1], se semnalează enunțul următor :

Dacă $a_n \geq 0$, $\alpha \geq 1$ și $\lim_{x \rightarrow 1} \frac{\sum a_n x^n}{\frac{1}{(1-x)^\alpha}} = 1$, atunci

(T)

$$\lim_{n \rightarrow \infty} \frac{a_0 + a_1 + \dots + a_n}{n^\alpha} = \frac{1}{\Gamma(\alpha + 1)}.$$

Cazul $\alpha = 1$ constituie enunțul unei teoreme datorate lui Hardy și Littlewood și în [1] se reproduce interesanta demonstrație dată de către J. Karamata acestei teoreme.

În această ordine de idei, vom demonstra propoziția :

V. Dacă (T) este adevărată pentru exponentul α , atunci este adevărată și pentru exponentul $\alpha + 1$.

Este clar că prin aceasta (T) este demonstrată pentru orice α natural și rămîne de considerat ulterior numai cazul $1 < \alpha < 2$.

Să presupunem că

$$\lim_{x \rightarrow 1} \frac{\sum a_n x^n}{\frac{1}{(1-x)^{\alpha+1}}} = 1,$$

de unde prin I

$$\lim_{x \rightarrow 1} \frac{\sum \frac{a_n}{n+1} x^{n+1}}{\frac{1}{\alpha} \frac{1}{(1-x)^\alpha} - \frac{1}{\alpha}} = \lim_{x \rightarrow 1} \frac{\sum \frac{a_n}{n+1} x^{n+1}}{\frac{1}{\alpha} \frac{1}{(1-x)^\alpha}},$$

ceea ce implică în virtutea ipotezei

$$\lim \frac{a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1}}{(n+1)^\alpha} = \frac{1}{\alpha \Gamma(\alpha+1)}.$$

Să punem: $u_n = a_0 + \frac{a_1}{2} + \dots + \frac{a_n}{n+1}$, $v_n = (n+1)^\alpha$,

de unde

$$a_0 = u_0, \quad a_1 = 2(u_1 - u_0), \dots, \quad a_n = (n+1)(u_n - u_{n-1})$$

și

$$\frac{a_0 + a_1 + \dots + a_n}{(n+1)^{\alpha+1}} = -\frac{u_0 + u_1 + \dots + u_{n-1}}{(n+1)^{\alpha+1}} + \frac{u_n}{(n+1)^\alpha}.$$

Dacă $w_n = u_0 + u_1 + \dots + u_n$,

$$\begin{aligned} \frac{w_n - w_{n-1}}{v_n - v_{n-1}} &= \frac{a_n}{(n+1)^{\alpha+1} - n^{\alpha+1}} = \frac{a_n}{n^{\alpha+1} \left[\left(1 + \frac{1}{n}\right)^{\alpha+1} - 1 \right]} = \\ &= \frac{a_n}{n^{\alpha+1} \left[\frac{\alpha+1}{n} + o\left(\frac{1}{n^2}\right) \right]} = \frac{a_n}{n^\alpha \left[\alpha+1 + o\left(\frac{1}{n}\right) \right]} \rightarrow \frac{1}{(\alpha+1) \alpha \Gamma(\alpha+1)}. \end{aligned}$$

În fine (Cesàro-Stolz)

$$\begin{aligned} \lim \frac{a_0 + a_1 + \dots + a_n}{(n+1)^{\alpha+1}} &= \frac{-1}{(\alpha+1) \alpha \Gamma(\alpha+1)} + \frac{1}{\alpha \Gamma(\alpha+1)} = \\ &= \frac{1}{(\alpha+1) \Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+2)}. \end{aligned}$$

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EXTENSION D'UNE DES RÈGLES D'HOSPITAL

(Résumé)

En remplaçant $\int_0^x f(x) dx$ par $F(x)$, construite à partir de $f(x)$ quelconque, on étend

la règle I à des cas plus généraux, $F(x)$ pouvant être univoque ou multivoque III, IV. Dans § 5 on démontre le résultat suivant :

V. Si (T) est vrai pour l'exposant α , alors il est vrai pour $\alpha + 1$ ($\alpha \geq 1$), où (T) est l'énoncé suivant, signalé par O. M a y e r [1]: Si $a_n \geq 0$, $\alpha \geq 1$ et

$$\lim \frac{\sum a_n x^n}{1/(1-x)^\alpha} = 1, \text{ alors } \lim \frac{a_0 + a_1 + \dots + a_n}{n^\alpha} = \frac{1}{\Gamma(\alpha + 1)}.$$

Si $\alpha = 1$, (T) [devient un théorème de H a r d y - L i t t l e w o o d, démontré par V; (T) est démontré pour α nombre naturel.

ON THE PERTURBATION FORMULAS OF GRÖBNER
AND ALEKSEEV ¹⁾

BY

G. WANNER and H. REITBERGER

1. Introduction

In the last years, the Perturbation Formulas of W. Gröbner and V. M. Alekseev have become a powerful tool in numerical computations as for qualitative discussions of the solutions of ordinary differential equations ([2], [13]).

This paper clears the connection between these two formulas (Sections 2 and 3). Next these formulas are generalized (Sections 4 and 5) to a formula for an arbitrary function of the solutions and to formulas with multiple integrals. In Sections 6 and 7 we give iteration methods, which include the methods of Picard, Gröbner-Knapp [8], Poincaré, Chen [5], as special case.

The last two sections sketch the application of the formulas in numerical computations and qualitative problems.

2. Gröbner's Formula

Consider the following differential equation:

$$(1) \quad \begin{aligned} y' &= f(x, y) = \hat{f}(x, y) + g(x, y), \\ y' &= \hat{f}(x, y), \end{aligned}$$

which may take on values in a Banach space E . We denote the solutions which take on the initial values x, y by $Y(X, x, y)$ respectively $\hat{Y}(X, x, y)$. Thus we have

¹⁾ This work was partially supported by the European Research Office, London, Contract No. JA 37-70-C-0249.

$$Y(x, x, y) = y, \quad \hat{Y}(x, x, y) = y.$$

We use the differential operators D , D_1 and D_2 which are defined by

$$D\Phi := \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} f(x, y),$$

$$D_1\Phi := \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \hat{f}(x, y),$$

$$D_2\Phi := (D - D_1)\Phi = \frac{\partial \Phi}{\partial y} g(x, y),$$

if $\Phi(x, y) : \mathbf{R} \times E \rightarrow F$ is differentiable, where F is any other Banach space.

We finally use the substitution rule

$$[D_2 \dots]_{\xi, \hat{Y}(\xi, x, y)}$$

which prescribes the substitutions

$$x \rightarrow \xi, \quad y \rightarrow \hat{Y}(\xi, x, y)$$

after differentiation.

Theorem 1. *If f , $\hat{f}$ and $\frac{\partial f}{\partial y}$ are continuous, then it holds that*

$$(2) \quad Y(X, x, y) = \hat{Y}(X, x, y) + \int_x^X [D_2 Y(X, \xi, y)]_{\xi, \hat{Y}(\xi, x, y)} d\xi.$$

The standard proof is as follows: Differentiation of $Y(X, x, y) = Y(X, \xi, Y(\xi, x, y))$ with respect to ξ gives for $\xi = x$

$$(3) \quad 0 = DY(X, x, y).$$

Similarly

$$\frac{\partial}{\partial \xi} Y(X, \xi, \hat{Y}(\xi, x, y)) = [D_1 Y(X, x, y)]_{\xi, \hat{Y}(\xi, x, y)}.$$

Subtracting this from (3) and integrating the result from x to X gives the stated result. Q.e.d.

3. Alekseev's Formula

In (2) we interchange Y and $\hat{Y}$, f and $\hat{f}$:

$$Y = \hat{Y} + \underbrace{\int \left[\frac{\partial}{\partial y} \hat{Y}(\hat{f} - f) \right]_{\xi, Y} d\xi}_{-D_2 \hat{Y}}.$$

This yields the following integral equation for the solution Y :

Theorem 2. If $\hat{f}$, f and $\frac{\partial \hat{f}}{\partial y}$ are continuous, then

$$(4) \quad Y(X, x, y) = \hat{Y}(X, x, y) + \int_x^X [D_2 \hat{Y}(X, \xi, y)]_{\xi, Y(\xi, x, y)} d\xi.$$

The variation of constants formula as a special case.

If $\hat{f}(x, y)$ is linear in y , then $\hat{Y}(X, \xi, y) = \hat{F}(X) \hat{F}^{-1}(\xi) y$, where $\hat{F}(X)$ is the fundamental system of solutions with $\hat{F}(x) = I$. Thus $\frac{\partial}{\partial y} \hat{Y}(X, \xi, y) = \hat{F}(X) \hat{F}^{-1}(\xi)$ and (4) is just the variation of constants formula for inhomogeneous linear differential equations. Hence the applications of the variation of constants formula to asymptotic theory, stability and control investigations or to the numerical treatment of stiff differential equations can be generalized to the nonlinear case.

If $\hat{f} = 0$, then (4) reduces to Picard's integral equation.

4. Generalizations

The above formulas can be generalized in the following way:

Theorem 3. Let $F(x, y) \in C^1$. Then it holds under the conditions of Theorem 1

$$(5) \quad F(X, Y(X, x, y)) = F(X, \hat{Y}(X, x, y)) + \int_x^X [D_2 F(X, Y(X, \xi, y))]_{\xi, \hat{Y}(\xi, x, y)} d\xi,$$

and under the conditions of Theorem 2

$$(6) \quad F(X, Y(X, x, y)) = F(X, \hat{Y}(X, x, y)) + \int_x^X [D_2 F(X, \hat{Y}(X, \xi, y))]_{\xi, Y(\xi, x, y)} d\xi.$$

Clearly, (5) and (6) coincide with (2) and (4) if $F(x, y) = y$.



P r o o f. Differentiate $F(X, Y(X, \xi, \hat{Y}(\xi, x, y)))$ with respect to ξ :

$$\frac{\partial}{\partial \xi} F(X, Y(X, \xi, \hat{Y}(\xi, x, y))) = \frac{\partial F}{\partial y}(X, Y(X, \xi, \hat{Y}(\xi, x, y))) [D_1 Y]_{\xi, \hat{Y}(\xi, x, y)}.$$

Next multiply (3) by $\frac{\partial}{\partial y} F(X, Y(X, \xi, \hat{Y}(\xi, x, y)))$ and subtract the two formulas. A subsequent integration from x to X then gives (5). Equation (6) is obtained from (5) by again interchanging Y and $\hat{Y}$. **Q. e. d.**

Remark. For $f = 1, \hat{f} = 0$ equations (5) and (6) are the „Fundamental Theorem of Integral Calculus”.

5. Iterated Equations

Still further integral equations are derived by iteration. Iterating (2) we obtain

$$(7) \quad \begin{aligned} Y(X, \xi_0, y) = & \hat{Y}(X, \xi_0, y) + \int_{\xi_0}^X [D_2 Y(X, \xi_1, y)]_{\xi_1, \hat{Y}(\xi_1, \xi_0, y)} d\xi_1 + \\ & + \int_{\xi_0}^X \int_{\xi_1}^X [D_2 [D_2 Y(X, \xi_2, y)]_{\xi_2, \hat{Y}(\xi_2, \xi_1, y)}]_{\xi_1, \hat{Y}(\xi_1, \xi_0, y)} d\xi_2 d\xi_1, \end{aligned}$$

which we shortly write as

$$(7') \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint D_2 [D_2 Y]_{\hat{Y}}.$$

Inserting equation (4) into (2) we obtain

$$(8) \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint [D_2 [D_2 \hat{Y}]_Y]_{\hat{Y}}.$$

It further holds that

$$(9) \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_Y,$$

which is due to **K. K u h n e r t**. Equation (9) is derived from (8) by putting $D_2 \hat{Y} = F$ and, after an interchange of order of integration, by using the formula

$$(10) \quad \int [D_2 F(X, Y)]_{\hat{Y}} = \int [D_2 F(X, \hat{Y})]_{\hat{Y}},$$

which follows from (5) and (6).

Again inserting (7), (8), (9) into (2) we obtain the equations

$$(11) \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}} + \iiint [D_2 [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}}]_{\hat{Y}},$$

$$(12) \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}} + \iiint [D_2 [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}}]_{\hat{Y}},$$

$$(13) \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}} + \iiint [D_2 [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}}]_{\hat{Y}}$$

and by the same argument as above

$$(14) \quad Y = \hat{Y} + \int [D_2 \hat{Y}]_{\hat{Y}} + \iint [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}} + \iiint [D_2 [D_2 [D_2 \hat{Y}]_{\hat{Y}}]_{\hat{Y}}]_{\hat{Y}}.$$

The general form of all possible equations is now visible.

6. Improvement of Arbitrary Approximate Solutions

If we insert an approximate solution $\tilde{Y}(X, \xi, y)$ into the integral in (2), we obtain a hopefully better approximation $\bar{Y}$ from the formula

$$(15) \quad \bar{Y}(X, x, y) = \hat{Y}(X, x, y) + \int_x^X [D_2 \tilde{Y}(X, \xi, y)]_{\xi, \hat{Y}(\xi, x, y)} d\xi.$$

An error expression may be obtained by subtracting (15) and (2). Concerning the order of $\bar{Y}$ the following is true:

Theorem 4. *If the order of $\hat{Y}$ and $\tilde{Y}$ are m and s respectively, then $\bar{Y}$ has order $m + s + 1$.*

Proof. From the assumptions we have

$$(16) \quad Y - \tilde{Y} = \frac{(X - \xi)^{s+1}}{(s+1)!} F(X, \xi, y),$$

$$(17) \quad [f(x, y) - \hat{f}(x, y)]_{\xi, \hat{Y}(\xi, x, y)} = G(\xi, x, y) \frac{(\xi - x)^m}{m!}$$

(cf. Wanner [21, p. 45]). Thus it follows from the definition of D_2 that

$$Y - \bar{Y} = \int_x^X \frac{(X - \xi)^{s+1}}{(s+1)!} \frac{(\xi - x)^m}{m!} F_y(X, \xi, y) G(\xi, x, y) d\xi.$$

With

$$\|F_y(X, \xi, y) G(\xi, x, y)\| \leq M$$

one has

$$\|Y - \bar{Y}\| \leq M \frac{(X - x)^{m+s+2}}{(m+s+2)!},$$

q. e. d.

The same can be done with each of the formulas (4), (7), (8), (9), (11), (12), (13), (14), ... as well, each giving a new corrector formula similar to (15).

7. Iteration Methods. Convergence

The equations (15), (16), ... are formulas, which give for any approximate solutions $\hat{Y}$ and $\tilde{Y}$ a hopefully better solution $\bar{Y}$. The repetition of this procedure leads to iteration methods. The iteration with respect to $\tilde{Y}$ leads to Poincaré's parameter expansion which contains the power series method as a special case. Its convergence for the analytic case is proved in [22, pp. 60–61]. Extensions of this result are due to K. Kuhnert.

Analogous iterations of (4), (9), (14), ... with respect to $\tilde{Y}$ lead to generalizations of iteration processes of Chen [5], which come out by putting $\hat{f}(x, y) = 0$ (see Kuhnert [16]).

Let us here consider the iteration with respect to $\hat{Y}$, thus the process

$$(18) \quad Y^{(r+1)} = Y^{(r)} + \int_x^X [D_2^{(r)} \tilde{Y}]_{\xi, Y^{(r)}} d\xi,$$

where

$$D_2^{(r)} \Phi = \frac{\partial \Phi}{\partial y} (f - f^{(r)}), \quad Y^{(r)'} = f^{(r)}(X, Y^{(r)}).$$

The convergence of this process is as follows:

Theorem 5. Assume in a domain B the estimations

$$(19) \quad \left\| \frac{\partial}{\partial y} (Y - \tilde{Y}) \right\| \leq \frac{M(X - \xi)^{(s+1)}}{(s+1)!},$$

$$(20) \quad \left\| \frac{\partial}{\partial X} \frac{\partial}{\partial y} (Y - \tilde{Y}) \right\| \leq \frac{N(X - \xi)^s}{s!},$$

$$(21) \quad \left\| f(X, Y^{(0)}) - f^{(0)}(X, Y^{(0)}) \right\| \leq \frac{H(X - x)^m}{m!},$$

which are possible whenever the approximate solutions $Y^{(0)}$ and $\tilde{Y}$ are of orders m and s respectively (cf. (16), (17)). Then it holds that

$$(22) \quad \left\| Y - Y^{(r)} \right\| \leq \frac{M K_{r-1} (X - x)^{r(s+1)+m+1}}{(r(s+1) + m + 1)!}$$

as long as $Y^{(r)}$ belongs to B . Here the constants K_r are recursively determined by

$$(23) \quad K_0 = H, \quad K_r = K_{r-1} \left(\frac{LM(X - x)}{r(s+1) + m + 1} + N \right)$$

and L is a Lipschitz constant for $f(x, y)$.

Proof. Write

$$f - f^{(r)} = S^{(r)}$$

then it holds that

$$(24) \quad \|S^{(r)}\| \leq \frac{K_r(X - x)^{r(s+1)+m}}{(r(s+1) + m)!},$$

what is seen by the following induction argument: $r = 0$ is condition (21). Assume the assertion for $r - 1$. Then

$$(25) \quad \begin{aligned} \|S^{(r)}\| &= \|f(X, Y^{(r)}) - f^{(r)}(X, Y^{(r)})\| \leq \\ &\leq \|f(X, Y^{(r)}) - f(X, Y)\| + \|f(X, Y) - f^{(r)}(X, Y^{(r)})\| \leq \\ &\leq L \|Y^{(r)} - Y\| + \|(Y^{(r)} - Y)'\|. \end{aligned}$$

Subtracting (18) from (2) we have

$$(26) \quad Y - Y^{(r)} = \int_x^X \left[\frac{\partial}{\partial y} (Y - \tilde{Y}) \right]_{\xi, Y^{(r-1)}} [S^{(r-1)}]_{\xi, Y^{(r-1)}} d\xi$$

and by (19) and the induction hypothesis

$$\begin{aligned} \|Y - Y^{(r)}\| &\leq M K_{r-1} \int_x^X \frac{(X - \xi)^{s+1}}{(s+1)!} \frac{(\xi - x)^{(r-1)(s+1)+m}}{((r-1)(s+1) + m)!} d\xi \leq \\ &\leq \frac{M K_{r-1} (X - x)^{r(s+1)+m+1}}{(r(s+1) + m + 1)!}. \end{aligned}$$

Differentiation of (26) with respect to X gives

$$(27) \quad (Y - Y^{(r)})' = \int_x^X \left[\frac{\partial}{\partial X} \frac{\partial}{\partial y} (Y - \tilde{Y}) \right] [S^{(r-1)}] d\xi,$$

since the partial derivative with respect to the upper limit of the integral vanishes for $\xi = X$:

$$\frac{\partial}{\partial y} (Y(X, X, y) - \tilde{Y}(X, X, y)) = 0.$$

Now by (20)

$$\| (Y - Y^{(r)})' \| \leq \frac{N K_{r-1} (X - x)^{r(s+1)+m}}{(r(s+1) + m)!}.$$

Thus by (25) and (27)

$$\| S^{(r)} \| \leq K_{r-1} \left(\frac{LM(X - x)}{r(s+1) + m + 1} + N \right) \frac{(X - x)^{r(s+1)+m}}{(r(s+1) + m)!},$$

which completes the induction together with the proof of (23). The estimation (22) is given by (27). Q. e. d.

Various other iterations are possible by the use of the other integral equations, some of them, however, do not give something new.

8. Special Cases

If one takes as the initial approximation Y the first s terms of the Taylor series expansion of Y

$$\tilde{Y}(X, \xi, y) = \sum_{\alpha=0}^s \frac{(X - \xi)^\alpha}{\alpha!} [D^\alpha y]_{\xi, y},$$

then one obtains the expression

$$(28) \quad \bar{Y}(X, x, y) = \hat{Y}(X, x, y) + \sum_{\alpha=0}^s \int_x^X \frac{(X - \xi)^\alpha}{\alpha!} [D_2 D^\alpha y]_{\xi, \hat{Y}(\xi, x, y)} d\xi.$$

Iterating this formula we obtain

$$Y^{(r+1)} = Y^{(r)} + \sum_{\alpha=0}^s \int_x^X \frac{(X - \xi)^\alpha}{\alpha!} [D_2^{(r)} D^\alpha y]_{\xi, Y^{(r)}} d\xi,$$

which is the iteration method of Gröbner-Knapp. In this case, by the use of the Bernoulli remainder term for the Taylor series expansion and by the use of formula (5) the following expression for the remainder of (28) can be derived

$$(29) \quad R = \int_x^X \frac{(X - \xi)^s}{s!} \{ [D^{s+1} y]_{\xi, Y(\xi)} - [D^{s+1} y]_{\xi, \tilde{Y}(\xi)} \} d\xi,$$

which is due to H. Knapp. It furnishes an easy error estimation if a Lipschitz constant L for the function $D^{s+1} y$ is known. It can further be used for a more simple convergence proof of the Gröbner-Knapp process (cf. Wanner [21, p. 61].

Putting $s = 0$, i. e., using the constant initial value for $\tilde{Y}$

$$\tilde{Y} = y,$$

leads to Picard's Iteration method ([22, p. 57]).

9. Application to Numerical Integration of ODE's

A method for the numerical integration of ordinary differential equations is based on formula (28). A possible choice for the approximate solution $\hat{Y}$ is the power series expansion

$$\hat{Y}(X, x, y) = \sum_{i=0}^m \frac{(X - x)^i}{i!} D^i y$$

so that

$$\hat{f}(X) = \sum_{i=0}^{m-1} \frac{(X - x)^i}{i!} D^{i+1} y.$$

A computer program for this method has been written together with H. Knapp, J. Gray and T. Szymanski [15]. The system of differential equations is given to the computer in Fortran notation, everything else is carried out automatically. The integrations are performed numerically by Gaussian quadrature. The program also computes the „Connection Matrix“, which consists of the derivatives of the solutions with respect to the initial values, and is particularly useful for boundary value problems, for the study of propagation of errors, and for stability investigations.

The experiences with this program are quite good. Numerical results can be found in Rall-Wanner [18], Knapp-Wanner [13], Wanner [21].

It is hoped that these formulas can also with success be applied to differential equations by pushing the stiffness into the approximate solution $\hat{Y}$.

Further, it is possible to apply the perturbation formulas to the integration of singular differential equations, extending thereby methods of D. L. Russell [18].

10. Application to Qualitative Questions

While in the preceding section the application of Gröbner's version of the perturbation formula to *quantitative* computations was mentioned, this section deals with the application of Alekseev's version to *qualitative* studies.

Assume that the stability or asymptotic behavior of the system

$$(30) \quad y' = \hat{f}(x, y)$$

is known. The question arises whether the „perturbed“ system

$$(31) \quad y' = \hat{f}(x, y) + g(x, y)$$

behaves in the same way.

Around 1900 Poincaré and Lyapunov have cleared up the special case, when (30) is linear and autonomous. In 1929 Perron examined this case again, his proof is based on the use of the „variation of constants formula“ ([6]). This is an integral equation which allows to estimate the solution of the perturbed system with the aid of estimates of the solution of the linear system and the nonlinear term.

The general case, i.e., if also the system (30) is nonlinear, can now analogously be treated with the aid of the Gröbner-Alekseev formula as a generalization of the variation of constants formula (cf. Section II.3.). This was initiated by Alekseev himself. The main results are listed below:

Brauer [2] shows the asymptotic stability of the trivial solution of (31) under two assumptions:

1. A somewhat more restrictive condition that asymptotic stability for the trivial solution of (30);

2. $g(x, y) = o(|y|)$ as $|y| \rightarrow 0$ uniformly in x .

In [3] Brauer proves among others that exponential asymptotic stability (cf. Hahn [9, p. 113] of the trivial solution of (30) and the above condition 2. for $g(x, y)$ imply the exponential asymptotic stability of the trivial solution of (31). (Other proofs for this result are given in Hahn [9] and Halanay [10]).

Strauss shows in [20], how certain boundedness conditions of the solutions of (30) pass over to those of (31). This result is due to Onuchic for the case of a perturbed linear system, the extension to the general case again was possible using the Alekseev formula.

In the recent paper [4] Brauer and Strauss treat perturbations of stable but not necessarily asymptotic stable systems. They give estimates for the growth of the perturbed solutions. For this reason the concept of „uniform stability in variation“ is introduced, which is more restrictive than the „integral stability“ of Vrkoc. This again is more restrictive than the usual „uniform stability“ (cf. Hahn [9, p. 173]).

In [3] Brauer started the study of the „asymptotic equivalence“ of the systems (30) and (31). Here two systems are called asymptotic equivalent, if there exists a 1-1 correspondence between the solutions, so that the difference of the corresponding solutions tend to zero as $t \rightarrow \infty$. Using the improper Alekseev integral equation, Brauer clears the first half of the equivalence problem, the extensive paper of Marlin and Struble [17] examine the other half as well.

Finally we want to mention that most of the given stability statements carry over as well to perturbed nonlinear differential-difference equations (retarded equations), since Hastings [11] has extended the Alekseev formula to such equations.

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ASUPRA FORMULELOR DE PERTURBAȚII ALE LUI GRÖBNER ȘI ALEKSEEV

(Rezumat)

Se precizează legătura dintre formulele de perturbații ale lui W. Gröbner și V. M. Alekseev, care au devenit în ultimii ani un instrument puternic atât pentru calculul numeric cât și pentru analiza calitativă a soluțiilor ecuațiilor diferențiale ordinare. În prealabil cele două formule sint generalizate într-o relație unică, după care se stabilesc metode mai generale de iterație decât cele ale lui Picard, Poincaré sau [8], [5].

ORDERS AND TOPOLOGIES INDUCED BY OTHERS AND BY MEANS OF MAPPINGS

BY

GH. COSTOVICI

1. Let ρ_1 and ρ_2 be binary relations defined on the sets P_1 and P_2 , respectively, and f a mapping for which P_1 is the domain and P_2 is the range. We introduce the relation ρ_3 on P_1 in the following way: $x\rho_3y \Leftrightarrow \Rightarrow [(f(x)\rho_2f(y) \text{ and } f(x) \neq f(y)) \text{ or } (f(x) = f(y) \text{ and } x\rho_1y)]$. We list some properties of ρ_3 .

I. Generally, ρ_1 and ρ_3 are incomparable in the order (reflexive, antisymmetric and transitive binary relation) $\leq$ defined on the set of the all binary relations defined on P_1 ; $\rho_\alpha \leq \rho_\beta \Leftrightarrow (x\rho_\alpha y \Rightarrow x\rho_\beta y)$.

II. If ρ_1 and ρ_2 are orders, then ρ_3 is order.

III. If ρ_1 and ρ_2 are total orders (an order ρ on a set M is total $\Leftrightarrow, x, y \in M, \exists x\rho y \text{ or } y\rho x$), then ρ_3 is total order. If $P_1 = P_2 = N$ = the set of all natural numbers, $\rho_1 = \rho_2$ = the increasing magnitude, and $f(x) =$ = the number of the natural divisors of x , we obtain the example from [1, p. 189].

IV. If P_1 is a group partially ordered by ρ_1 , P_2 is a group partially ordered by ρ_2 , and f is a homomorphism of groups, then P_1 becomes group partially ordered by ρ_3 . The foregoing are easy to verify. This first part of this note and [2] inspired what follows.

2. Let τ_1 be a topology defined on the set P_1 , τ_2 a T_0 -topology (a topology τ is $T_0 \Leftrightarrow$ distinct points have not identical closures in τ) defined on P_2 , and f a mapping for which P_1 is the domain and P_2 is the range. Then τ_3 , defined in the following way:

$\overline{A}^{\tau_3} = \{x \in P_1 : \exists a \in A \text{ and } [(f(x) \in \overline{f(a)}^{\tau_2} \text{ and } f(x) \neq f(a)) \text{ or } (f(x) = f(a) \text{ and } x \in \overline{a}^{\tau_1})]\}$, is a topology on P_1 ($\overline{A}$ denots the closure of A in the topology τ_α). We have:

I. Generally, τ_1 and τ_3 are incomparable in the order $\leq$ defined on the set of the all topologies defined on P_1 ; $\tau_\beta \leq \tau_\alpha \Leftrightarrow \bar{A}^{\tau_\beta} \subset \bar{A}^{\tau_\alpha}$.

II. τ_3 is discrete topology ($\bigcup_{\gamma \in \Gamma} \bar{X}_\gamma = \bigcup_{\gamma \in \Gamma} x_\gamma$, Γ arbitrary).

III. If τ_1 is T_0 -topology, then τ_3 is T_0 -topology.

IV. If P_1 and P_2 are groups, τ_1 and τ_2 are compatible with the operation, respectively, and f is homomorphism of groups, then τ_3 is compatible with the operation (the topology τ_α is compatible with operation of

a group $G \Leftrightarrow \bar{AB}^{\tau_\alpha} \subseteq \overline{AB}^{\tau_\alpha}$, A and B contained in G). The statements in the second part of this note are not so difficult to verify.

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ORDINI ȘI TOPOLOGII INDUSE DE ALTELE CU AJUTORUL APLICAȚIILOR

(Rezumat)

Din relații binare și topologii date, se obțin, cu ajutorul aplicațiilor, relații și topologii noi, care au unele proprietăți legate de ale celor inițiale.

ASUPRA UNEI CLASE DE ALGEBRE DE ORDINUL ȘASE

DE

I. ENESCU

1. În [1] s-a arătat că, pentru oricare ordin finit par, superior lui patru, există algebre asociative, comutative și cu unitate principală, indecompozabile, cu scalari reali, care să aibă structura de sumă (nedirectă) dintre algebra K a numerelor complexe și radicalul N , care să fie diferit de algebra nulă. Tot acolo, se pune problema determinării exhaustive, pentru un ordin dat, a tipurilor neizomorfe de astfel de algebre.

În această Notă, ne propunem să răspundem acestei chestiuni, pentru algebre de ordinul șase.

2. Fie deci o algebră reală de ordin șase, de forma

$$(1) \quad A = K + N,$$

unde $\text{ordin } N = 4$, $\text{ordin } K = 2$, radicalul N fiind diferit de algebra nulă.

Pentru a putea cerceta toate tipurile posibile de algebre (1), vom utiliza tipurile neizomorfe de algebre nilpotente comutative și asociative de ordinul patru, determinate complet în [2]. Vom nota prin 1 și i ($i^2 = -1$), o bază pentru K , 1 fiind unitatea principală a algebrei A . De asemenea, notăm $n_j = \text{ordin } N^j$, ($j = 1, \dots, 5$) și $\delta = \text{ordin } N - \text{ordin } N^2$.

Radicalul N , ca algebră reală nilpotentă, asociativă și comutativă de ordin patru, prezintă [2] următoarele tipuri neizomorfe:

a) $n_1 = 4$, $n_2 = 3$, $n_3 = 2$, $n_4 = 1$, $n_5 = 0$, $\delta = 1$, un tip dat prin tabela

$$(2) \quad \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & u_2 & u_3 & u_4 & 0 \\ u_2 & u_3 & u_4 & 0 & 0 \\ u_3 & u_4 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 \end{array},$$

b) $n_1 = 4, n_2 = 2, n_3 = 1, n_4 = 0$; două tipuri, date respectiv prin tabelele :

(3)

	u_1	u_2	u_3	u_4
u_1	$-u_4$	0	0	0
u_2	0	u_3	u_4	0
u_3	0	u_4	0	0
u_4	0	0	0	0

(4)

	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	u_3	u_4	0
u_3	0	u_4	0	0
u_4	0	0	0	0

c) $n_1 = 4, n_2 = 2, n_3 = 0$; patru tipuri, anume :

(5)

	u_1	u_2	u_3	u_4
u_1	u_3	u_4	0	0
u_2	0	0	0	0
u_3	0	0	0	0
u_4	0	0	0	0

(6)

	u_1	u_2	u_3	u_4
u_1	u_3	0	0	0
u_2	0	u_4	0	0
u_3	0	0	0	0
u_4	0	0	0	0

(7)

	u_1	u_2	u_3	u_4
u_1	u_3	$u_3 + u_4$	0	0
u_2	$u_3 + u_4$	u_4	0	0
u_3	0	0	0	0
u_4	0	0	0	0

(8)

	u_1	u_2	u_3	u_4
u_1	u_3	u_3	0	0
u_2	u_3	u_4	0	0
u_3	0	0	0	0
u_4	0	0	0	0

d) $n_1 = 4, n_2 = 1, n_3 = 0$; cinci tipuri și anume :

(9)

	u_1	u_2	u_3	u_4
u_1	u_4	0	0	0
u_2	0	u_4	0	0
u_3	0	0	u_4	0
u_4	0	0	0	0

(10)

	u_1	u_2	u_3	u_4
u_1	u_4	0	0	0
u_2	0	u_4	0	0
u_3	0	0	$-u_4$	0
u_4	0	0	0	0

$$(11) \quad \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & u_4 & 0 & 0 & 0 \\ u_2 & 0 & u_4 & 0 & 0 \\ u_3 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 \end{array} , \quad (12) \quad \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & -u_4 & 0 & 0 & 0 \\ u_2 & 0 & u_4 & 0 & 0 \\ u_3 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 \end{array} ,$$

$$(13) \quad \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & u_4 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 & 0 \\ u_3 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 \end{array} .$$

Apar deci ca posibile, 12 tipuri de algebre A de ordin șase, cu structura (1), în care radicalul N are, respectiv, una din formele (2) ... (13).

3. Se pune acum problema asociativității pentru algebrele A de mai sus. Se observă imediat că trebuie cercetată numai asociativitatea produselor de forma $i(u_j u_k)$ și $i(iu_j)$, ($j, k = 1, \dots, 4$).

Notăm, pentru orice algebră A , produsele iu_j , prin respectiv :

$$(14) \quad iu_1 = \sum_1^4 \alpha_j u_j, \quad iu_2 = \sum_1^4 \beta_j u_j, \quad iu_3 = \sum_1^4 \gamma_j u_j, \quad iu_4 = \sum_1^4 \delta_j u_j.$$

Cazul a). Condițiile $i^2 u_i = i(iu_i)$, $i(u_1 u_1) = (iu_1)u_1$, $i(u_2 u_2) = (iu_2)u_2$ și $i(u_3 u_3) = (iu_3)u_3$ conduc la $\alpha_2^2 = -1$, dovedind că tipul a) nu există.

La aceeași concluzie se ajunge și pentru tipul dat de (3), unde, utilizînd condițiile $i(u_1 u_1) = (iu_1)u_1$ și $i^2 u_4 = i(iu_4)$, se găsește $\delta_4^2 = -1$. Tot în cazul b), pentru tipul dat de (4), condițiile $i(u_2 u_2) = (iu_2)u_2$, $i^2 u_3 = i(iu_3)$ și $i(u_2 u_3) = (iu_2)u_3$, conduc la $\gamma_5^2 = -1$, stabilind inexistența algebrei respective.

Cazul c). Condițiile $i(u_1 u_1) = (iu_1)u_1$, $i^2 u_3 = i(iu_3)$, $(iu_2)u_2 = i(u_2 u_2)$ și $(iu_2)u_3 = i(u_2 u_3)$, impuse tipului (5), conduc la $\gamma_3^2 = -1$, ceea ce arată că acest tip nu există.

La fel, pentru tipul dat de (6), condițiile $(iu_1)u_1 = i(u_1 u_1)$ și $i^2 u_3 = i(iu_3)$, dau $\gamma_3^2 = -1$, conducînd la aceeași concluzie ca și în cazul precedent.

Prin procedee similare, se constată că tipul (8) din cazul c) precum și toate tipurile ((9) ... (12)), cazului d), nu există.

Rămîne în discuție tipul dat de (7). Existența sa este asigurată, observînd că există algebre cu proprietățile cerute în problema de față și acestea sînt polinomialele generate de polinomul $(\lambda^2 + 1)^3$. Pe de altă

parte, se știe că [3] acestea sînt caracterizate prin $\delta = 2$. Cum, exceptînd tipul (7), toate cazurile cu $\delta = 2$ nu se pot prezenta, rezultă imediat că tipul (7) există și, pînă la izomorfism, este unicul tip de asemenea algebră.

Rezultatele stabilite, dovedesc următoarea

Teoremă. *Singurul tip de algebră de ordin șase cu structura (1), este acela reprezentat de algebra polinomială generată de polinomul $(\lambda^2 + 1)^3$.*

4. Teorema de mai sus sugerează tratarea aceleiași chestiuni pentru algebre de ordin par superior lui șase. Natural, aceasta presupune determinarea exhaustivă a tipurilor neizomorfe de algebre nilpotente de ordinul par imediat inferior celui considerat.

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SUR UNE CLASSE D'ALGÈBRES D'ORDRE SIX

(Résumé)

On pose le problème de déterminer tous les types non-isomorphes d'algèbres d'ordre six qui ont la structure (1), où K est le corps complexe et le radical N est une algèbre nilpotente qui n'est pas l'algèbre nulle. On trouve qu'il existe un seul type, donné par l'algèbre polynomiale générée par le polynôme $(\lambda^2 + 1)^3$.

ON THE ASYMPTOTIC EQUIVALENCE OF RECTILINEAR MOTION

BY

G. LADAS¹⁾ and C. P. TSOKOS

1. The differential equation which describes the ideal rectilinear motion of a body, B , moving without acceleration, that is, the algebraic sum of all external forces acting on B is zero, is given by

$$(1.1) \quad x''(t) = 0, \quad \left('' \equiv \frac{d^2}{dt^2} \right).$$

Thus, the displacement equation is given by

$$(1.2) \quad x(t) = ut + x_0,$$

where u is its constant velocity and x_0 the initial position of B . Of course, such a description of a physical phenomenon is ideal. On the body, B , act external forces such as air resistance, friction, etc., and its future behavior depends in many circumstances not only on its present state but also on its past history, [3]. The actual description of the behavior of the body, B , is given by a functional differential equation of order two of the form

$$(1.3) \quad y''(t) = f(t, y(t), y(g_1(t)), \dots, y(g_k(t))),$$

where

$$f \in C([0, \infty) \times R^{k+1}, R), \quad g_j \in C([0, \infty), R), \quad (j = 1, \dots, K),$$

with C denoting the class of continuous mappings from $[0, \infty)$ into R . This functional differential equation shows that the rate of change of the velocity of B is influenced by hereditary effects.

¹⁾ The author acknowledges the support of a summer faculty fellowship from the University of Rhode Island.

Bellman [1] has studied extensively the linear form of equation (1.3), without retardations, that is, the case

$$f(t, y(t), y(g_1(t)) \dots, y(g_k(t))) \equiv p(t)y(t).$$

Quite recently Cohen [2] has investigated the nonlinear form of equation (1.3) without retardations,

$$f(t, y(t), y(g_1(t)), \dots, y(g_k(t))) \equiv F(t, y(t)).$$

The aim of this note is to prove that the solutions of the functional differential equation (1.3) behave similarly to the solutions of equation (1.1) for sufficiently large values of the real argument t .

2. With respect to the functional differential equation we shall assume that

$$(2.1) \quad g_j(t) \leq t \quad \text{and} \quad g_j(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty$$

for $j = 0, 1, 2, \dots, k$.

Theorem. Suppose that

$$(2.2) \quad \begin{aligned} |f(t, x_0, x_1, \dots, x_k)| &\leq \sum_{j=0}^k P_j(t) |x_j|, \\ (t, x_0, x_1, \dots, x_k) &\in [0, \infty) \times R^{k+1}, \end{aligned}$$

where $p_j g_j \in L_1[0, \infty)$, ($j = 0, 1, 2, \dots, k$) with $g_0(t) \equiv t$.

Then the solution of (1.3) are asymptotic to the solutions of (1.1) as $t \rightarrow \infty$.

Proof. Here we shall employ certain methods of Bellman [1] and a technique recently employed by Wed [5]. Choose $t_0 \geq 1$ large enough so that

$$g_j(t) \geq t_0 \quad \text{for} \quad t \geq t_0, \quad (j = 1, 2, \dots, k).$$

Such a choice of the real argument is possible because of (2.1). Integrating the functional differential equation (1.3) twice from t_0 to t we obtain

$$(2.3) \quad \begin{aligned} y(t) &= [y(t_0) + (t - t_0)y'(t_0)] + \\ &+ \int_{t_0}^t (t-s)f(s, y(s), y(g_1(s)), \dots, y(g_k(s))) ds. \end{aligned}$$

From (2.3) and in view of (2.2) we get for $t \geq t_0 + 1$ the estimate

$$\begin{aligned}
& |y(t)| \leq |y(t_0)| + (t - t_0) |y'(t_0)| + \\
& + \int_{t_0}^t (t - s) |f(s, y(s), y(g_1(s)), \dots, y(g_k(s)))| ds \leq [|y(t_0)| + |y'(t_0)|] (t - t_0) + \\
& + (t - t_0) \int_{t_0}^t \sum_{j=0}^k p_j(s) |y(g_j(s))| ds \leq \\
& \leq t \left[D + \int_{t_0}^t \sum_{j=0}^k p_j(s) |y(g_j(s))| ds \right],
\end{aligned}$$

where D is a constant, $D = |y(t_0)| + |y'(t_0)|$ and $g_0(t) \equiv t$. Thus, we have

$$(2.4) \quad |y(t)| \leq t \left[D + \int_{t_0}^t \sum_{j=0}^k p_j(s) |y(g_j(s))| ds \right].$$

Define

$$(2.5) \quad m(t) = D + \int_{t_0}^t \sum_{j=0}^k p_j(s) |y(g_j(s))| ds.$$

Differentiating (2.5) we have

$$(2.6) \quad m'(t) \leq \sum_{j=0}^k p_j(t) |y(g_j(t))|.$$

However, from equations (2.4) and (2.5) and the increasing character of $m(t)$ we have

$$(2.7) \quad |y(g_j(t))| \leq g_j(t) m(t), \quad (j = 1, 2, \dots, k),$$

and in view of (2.7) and inequality (2.6) we get

$$(2.8) \quad m'(t) \leq \left\{ \sum_{j=0}^k p_j(t) g_j(t) \right\} m(t)$$

and

$$(2.9) \quad m(t_0) = D.$$

From inequality (2.8) and (2.9) and the fundamental inequality [1], we obtain

$$(2.10) \quad m(t) \leq D \exp \sum_{j=0}^k \left\{ \int_{t_0}^t p_j(s) g_j(s) ds \right\}.$$

In view of (2.4) and (2.10) we have

$$|y(t)| \leq t D \exp \sum_{j=0}^k \left\{ \int_{t_0}^t p_j(s) g_j(s) ds \right\}$$

and since $p_j g_j \in L_1[0, \infty)$ for $j = 0, 1, \dots, k$ we get

$$\frac{|y(t)|}{t} \leq D \exp \int_{t_0}^t \sum_{j=0}^k p_j(s) g_j(s) ds \leq D \exp \int_{t_0}^{\infty} \sum_{j=0}^k p_j(s) g_j(s) ds = \tilde{D} < \infty,$$

where $\tilde{D}$ is a constant. Thus, we have shown

$$(2.11) \quad |y(t)| \leq \tilde{D}t.$$

Integrating the functional equation (1.3) once from t_0 to t we get

$$(2.12) \quad y'(t) = y'(t_0) + \int_{t_0}^t f(s, y(s), y(g_1(s)), \dots, y(g_k(s))) ds.$$

We shall prove that $\lim_{t \rightarrow \infty} y'(t)$ exists and is finite. It suffices to show that the integral part of equation (2.12) converges as $t \rightarrow \infty$. In view of inequalities (2.11) and (2.2) this assertion is true. Indeed,

$$\begin{aligned} \int_{t_0}^t |f(s, y(s), y(g_1(s)), \dots, y(g_k(s)))| ds &\leq \int_{t_0}^t \sum_{j=0}^k p_j(s) |y(g_j(s))| ds \leq \\ &\leq \int_{t_0}^t \sum_{j=0}^k p_j(s) \tilde{D} g_j(s) ds \leq \tilde{D} \int_{t_0}^{\infty} \sum_{j=0}^k p_j(s) g_j(s) ds < \infty. \end{aligned}$$

Therefore, since $\lim_{t \rightarrow \infty} y'(t)$ exists we conclude that

$$y(t) \sim ut + x_0$$

and the theorem is proven.

Remark 2.0. One can always construct a solution of the functional differential equation (1.3) with $u \neq 0$. We simply have to choose $y'(t_0)$ in (2.12) in such a way so that

$$y'(t_0) - \tilde{L} \int_{t_0}^t \sum_{j=0}^k p_j(s) g_j(s) ds \neq 0.$$

Remark 2.1. The solutions of equation (1.3) are nonoscillatory, that is, they have no more than one zero solution in $[0, \infty)$.

Remark 2.2. In the course of the proof of the theorem we have shown that all the solutions $y(t)$ of the functional differential equations are such that

$$|y(t)| \leq tk, \quad \text{for large } t,$$

where k is some constant.

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ASUPRA ECHIVALENȚEI ASIMPTOTICE A MIȘCĂRII RECTILINII

(Rezumat)

Se demonstrează că soluțiile ecuației diferențiale funcționale (1.3) se comportă în mod similar cu soluțiile ecuației (1.1), pentru valori suficient de mari ale argumentului real t .

ON THE STABILITY OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $\varphi'(x) = f(x, \varphi(A_x))$

BY

DAN BORȘ

1. Let X and Y be nonvoid sets, T a tribe of subsets of X which contains X and $T \times Y$ the cartesian product of pairs (A, x) where $A \in T$ and $x \in Y$. The tribe T is an associative and commutative ring with the unity X under the laws $(A, B) \rightarrow A \Delta B$ and $(A, B) \rightarrow A \cap B$ (symmetric difference and intersection). If Y is a real associative and commutative algebra with the unity e under the laws $(x, y) \rightarrow x + y$, $(\alpha, x) \rightarrow \alpha x$ and $(x, y) \rightarrow xy$ then [3] the set $T \times Y$ is a real associative and commutative quasialgebra with the unity (X, e) under the laws $(A, x) + (B, y) = (A \Delta B, x + y)$, $\alpha(A, x) = (A, \alpha x)$, $(A, x)(B, y) = (A \cap B, xy)$.

2. Suppose X is an open subspace of a metric space M with the metric d and (X, T, μ) is a measure space where $\mu: T \rightarrow R^1$ is a positive measure. The measure μ satisfies the properties $\mu(\emptyset) = 0$ and $\mu(A \Delta B) \leq \mu(A) + \mu(B)$ for any two sets A, B belonging to T . The set $T \times T$ is a ring under the laws $(A, B) \circ (A', B') = (A \Delta A', B \Delta B')$, $(A, B) * (A', B') = (A \cap A', B \cap B')$ and the mapping $\tilde{\mu}: T \times T \rightarrow R^1$ defined by $\tilde{\mu}(A, B) = \mu(A) + \mu(B)$ satisfies the properties $\tilde{\mu}(\emptyset, \emptyset) = 0$, $\tilde{\mu}((A, B) \circ (A', B')) \leq \tilde{\mu}(A, B) + \tilde{\mu}(A', B')$ for any two pairs $(A, B), (A', B')$ belonging to $T \times T$. The set T , respectively $T \times T$, is a pseudo-metric space with the pseudo-metric $d(A, B) = \mu(A \Delta B)$, respectively $d((A, B), (A', B')) = d(A, A') + d(B, B')$.

Proposition 1. *The symmetric difference, the intersection and the pseudo-metric d are continuous.*

It means that for any $\varepsilon > 0$ and for $(A_0, B_0) \in T \times T$, there exists a $\delta = \delta(\varepsilon, (A_0, B_0)) > 0$ such that for any $(A, B) \in T \times T$ the inequality $\tilde{\mu}(A, B) \circ (A_0, B_0) < \delta$ implies $\mu((A \Delta B) \Delta (A_0 \Delta B_0)) < \varepsilon$, respectively $\mu(A \cap$

$\cap B) \Delta(A_0 \cap B_0)) < \varepsilon$; for d , the inequality $\tilde{d}((A, B), (A_0, B_0)) < \delta$ implies $|d(A, B) - d(A_0, B_0)| < \varepsilon$.

3. Suppose the real algebra Y to be normed. Then the mapping $(A, x) \rightarrow \|(A, x)\|$ of $T \times Y$ into R^1 defined by

$$(1) \quad \|(A, x)\| = \mu(A) + \|x\|$$

has the following properties:

$$(1^\circ) \quad \|(\emptyset, \theta)\| = 0;$$

$$(2^\circ) \quad \|(A, x) + (B, y)\| \leq \|(A, x)\| + \|(B, y)\|$$

for any two pairs (A, x) and (B, y) belonging to $T \times Y$;

$$(3^\circ) \quad |\alpha| \leq 1 \text{ implies } \|\alpha(A, x)\| \geq |\alpha| \|(A, x)\|$$

for any (A, x) belonging to $T \times Y$ and α to R^1 ;

$$(4^\circ) \quad |\alpha| \geq 1 \text{ implies } \|\alpha(A, x)\| \leq |\alpha| \|(A, x)\|$$

for any (A, x) belonging to $T \times Y$ and α to R^1 ;

(5°) The following two conditions are equivalent:

$$(a) \quad \|(A, x)(B, y)\| \leq \|(A, x)\| \|(B, y)\|,$$

$$(b) \quad \mu(A \cap B) - \mu(A)\mu(B) \leq \mu(A)\|y\| + \mu(B)\|x\| + \|x\|\|y\| - \|xy\|.$$

It is obvious that each of the conditions

$$(b_1) \quad \mu(A) = 0 \text{ or } \mu(B) = 0 \text{ or } \mu(A \cap B) = 0,$$

$$(b_2) \quad \|x\| \geq 1 \text{ or } \|y\| \geq 1$$

implies (a).

The mapping defined by (1) is to be called a „norm on $T \times Y$ “.

It is obvious how to define the laws of the real quasi-algebra $(T \times Y) \times (T \times Y)$; the „norm on $(T \times Y) \times (T \times Y)$ “ will be

$$|||((A, x), (B, y))||| = \|(A, x)\| + \|(B, y)\|;$$

the pseudo-metric on $T \times Y$, respectively on $(T \times Y) \times (T \times Y)$ will be

$$d((A, x), (B, y)) = \mu(A \Delta B) + \|x - y\|,$$

respectively

$$\begin{aligned} d(((A, x), (B, y)), ((A', x'), (B', y'))) &= \\ &= d((A, x), (A', x')) + d((B, y), (B', y')). \end{aligned}$$

Proposition 2. *The laws of the real quasialgebra $T \times Y$ and its „norm“ are continuous.*

To prove that we need the „norm“

$$\|(\alpha, (A, x))\|_1 = |\alpha| + \|(A, x)\|$$

on the cartesian product $R^1 \times (T \times Y)$.

4. Let $\varphi: T \rightarrow R^1$ be a countably additive and continuous mapping, $F: X \rightarrow T'$ a continuous bijection denoted by $F(x) = A_x$ for $x \in X$, $A_x \in T' \subseteq T$ and $u: X \rightarrow R^1$ the mapping defined by the composition $\varphi|_{T'} \circ F$. The mapping u is then continuous and we denote it by $\varphi \circ F$.

Let f be a continuous, bounded and lipschitzian mapping with respect to the second argument (L being the lipschitzian constant) which applies an open subset D of $M \times R^1$ into R^1 .

Under these conditions [1] and $\mu(X) < L^{-1}$ [4], the differential equation

$$(2) \quad \varphi'(x) = f(x, \varphi(A_x)), \quad x \in X, \quad A_x \in T' \subseteq T$$

has a unique solution $\varphi: T' \rightarrow R^1$ with arbitrary given values on any intersection of the sets belonging to T' with a given zero measure set A_0 of T ; that means there exists a continuous and bounded mapping $u_0: X \rightarrow R^1$ defined by

$$(3) \quad u_0(x) = \varphi(A_x \cap A_0), \quad x \in X;$$

we have called the relations (3) the initial conditions of the equation (2) and we have denoted the solution φ which satisfies (2) and (3) by

$$(4) \quad \varphi = \varphi(A_x, A_0, u_0).$$

5. *I_0 -stability.* We call unperturbed the solution (4) of the differential equation (2).

Let I be the ideal of zero measure sets of T and $I_0 = I \cap T'$.

For given $r > 0$ and $N_0 \in I_0$, a solution $\psi = \psi(A_x, N_0, v_0)$ of the differential equation (2) is called (N_0, r) -perturbed if $\delta(u_0, v_0) < r$ [4].

The unperturbed solution (4) of the differential equation (2) is called a I_0 -stable solution if for any $\varepsilon > 0$ and $N_0 \in I_0$, there exists $\eta = \eta(N_0, \varepsilon) > 0$ such that for any (N_0, η) -perturbed solution $\psi = \psi(A_x, N_0, \eta)$ of the differential equation (2) we have

$$|\varphi(A_x, A_0, u_0) - \psi(A_x, N_0, v_0)| < \varepsilon.$$

Proposition 3. *The following conditions are equivalent:*

(a) *The unperturbed solution (4), of the differential equation (2) is a I_0 -stable solution.*

(b) *There exists a continuous strictly increasing surjection $K: I_0 \times (0, +\infty) \rightarrow (0, +\infty)$ with $\lim_{\alpha \rightarrow 0} K(A_0, \alpha) = 0$ such that for any $N_0 \in I_0$, for any $(N_0, K(N_0, \alpha))$ -perturbed solution $\psi = \psi(A_x, N_0, v_0)$ of the differential equation (2), for any $A_x \in T'$ and for any right inverse k of K [2], we have*

$$|\varphi(A_x, A_0, u_0) - \psi(A_x, N_0, v_0)| < (\text{pr}_2 \circ k)(\delta(u_0, v_0)).$$

Proof. Let $\eta_0 = \eta_0(N_0, \varepsilon) > 0$ be the supremum of all positive real numbers η satisfying the definition of I_0 -stability of the solution (4). Therefore $\delta(u_0, v_0) < \eta_0$ implies

$$|\varphi(A_x, A_0, u_0) - \psi(A_x, N_0, v_0)| < \varepsilon$$

and $\delta(u_0, v_0) < \eta_1$, $\eta_0 < \eta_1$ implies the existence of a set N'_0 belonging to I_0 and a continuous and bounded mapping $v'_0: X \rightarrow R'$ such that

$$|\varphi(A_x, A_0, u_0) - \psi(A_x, N'_0, v'_0)| \geq \varepsilon.$$

Let $K: I_0 \times (0, +\infty) \rightarrow (0, +\infty)$ be a continuous strictly increasing surjection with the properties $\lim_{\alpha \rightarrow 0} K(A_0, \alpha) = 0$ and $K(N_0, \varepsilon) < \eta_0$ ($N_0 \varepsilon$) for any $(N_0, \varepsilon) \in I_0 \times (0, +\infty)$.

Then $\eta = K(N_0, \varepsilon)$ and therefore $\varepsilon = (\text{pr}_2 \circ k)(\eta)$ for any right inverse k of K . The condition (b) holds because $\delta(u_0, v_0) < \eta$, that is (a) implies (b).

Conversely, let ε be an arbitrary positive real number. By definition of the mapping K it follows that we can choose $\varepsilon = (\text{pr}_2 \circ k)(\eta)$ for $\delta(u_0, v_0) < \eta$. Any k being bijective we deduce

$$\eta = (\text{pr}_2 \circ k)^{-1}(\varepsilon) = k^{-1}(\text{pr}_2(\varepsilon)) = k^{-1}(N_0, \varepsilon) = K(N_0, \varepsilon)$$

for any $N_0 \in I_0$. The condition (a) holds and therefore (b) implies (a). Q.e.d.

6. Stability with respect to a chain. For any $N_0 \in I_0$, consider the set $\mathcal{L}_{N_0}$ of the chains L_{N_0} included in T' such that $N_0 \subseteq A$ for any $A \in L_{N_0}$. For $N_0, N'_0 \in I_0$ the chains L_{N_0} and $L_{N'_0}$ are called g -equivalent if g is a bijection of L_{N_0} on $L_{N'_0}$ and $N'_0 = g(N_0)$; we denote by $L_{g(N_0)}$ a chain such that L_{N_0} and $L_{g(N_0)}$ are g -equivalent.

The unperturbed solution (4) is called a stable solution with respect to the chain L_{A_0} if for any $\varepsilon > 0$ and for any $A_x \in L_{A_0}$ there exists $\eta = \eta(A_x, \varepsilon) > 0$ and a chain $L_{g(A_0)}$, then for any $(g(A_0), \eta)$ -perturbed solution $\psi = \psi(B_y, g(A_0), v_0)$ of the differential equation (2), with arbitrary $B_y \in T'$, we have

$$\|(A_x, \varphi(A_x, A_0, u_0)) - (g(A_x), \psi(g(A_x), g(A_0), v_0))\| < \varepsilon.$$

Proposition 4. The following conditions are equivalent:

(a) The unperturbed solution (4) of the differential equation (2) is stable with respect to the chain L_{A_0} .

(b) There exists a continuous strictly increasing surjection $K: L_{A_0} \times (0, +\infty) \rightarrow (0, +\infty)$ with $\lim_{\alpha \rightarrow 0} K(A_0, \alpha) = 0$ and a chain $L_{g(A_0)}$ such that for any $(g(A_0), K(A_x, \varepsilon))$ -perturbed solution $\psi = \psi(B_y, g(A_0), v_0)$ with arbitrary $B_y \in T'$, for any right inverse k of K and for any $A_x \in L_{A_0}$ we have

$$\|(A_x, \varphi(A_x, A_0, u_0)) - (g(A_x), \psi(g(A_x), g(A_0), v_0))\| < (\text{pr}_2 \circ g)(\delta(u_0, v_0)).$$

Proof. Let $\eta_0 = \eta_0(A_x, \varepsilon) > 0$ be the supremum of all positive real numbers η satisfying the definition of the stability of φ with respect to the chain L_{A_0} . Therefore $\delta(u_0, v_0) < \eta_0$ implies

$$\|(A_x, \varphi(A_x, A_0, u_0)) - (g(A_x), \psi(g(A_x), g(A_0), v_0))\| < \varepsilon$$

and $\delta(u_0, v_0) < \eta_1$, $\eta_0 < \eta_1$ implies the existence of a chain $L_{g'(A_0)}$, and a continuous and bounded mapping $v'_0: X \rightarrow R^1$ such that

$$\|(A_x, \varphi(A_x, A_0, u_0)) - (g'(A_x), \psi(g'(A_x), g'(A_0), v'_0))\| \geq \varepsilon.$$

Let $K: L_{A_0} \times (0, +\infty) \rightarrow (0, +\infty)$ be a continuous strictly increasing surjection with the properties $\lim_{\alpha \rightarrow 0} K(A_0, \alpha) = 0$ and $K(A_x, \varepsilon) < \eta_0(A_x, \varepsilon)$ for any $(A_x, \varepsilon) \in L_{A_0} \times (0, +\infty)$. Then $\eta = K(A_x, \varepsilon)$ and therefore $\varepsilon = (\text{pr}_2 \circ k)(\eta)$ for any right inverse k of K . The condition (b) holds because $\delta(u_0, v_0) < \eta$, that is (a) implies (b).

Let ε be an arbitrary positive real number. By definition of the mapping K it follows we can choose $\varepsilon = (\text{pr}_2 \circ k)(\eta)$ for $\delta(u_0, v_0) < \eta$. Any k being bijective we deduce $\eta = K(A_x, \varepsilon)$ for any $A_x \in L_{A_0}$. The condition (a) holds that is (b) implies (a). Q. e. d.

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ASUPRA STABILITĂȚII SOLUȚIILOR ECUAȚIEI DIFERENȚIALE

$$\varphi'(x) = f(x, \varphi(A_x))$$

(Rezumat)

Definindu-se o „normă“ pe cuasialgebra $T \times Y$ se consideră apoi pentru soluțiile ecuației diferențiale $\varphi'(x) = f(x, \varphi(A_x))$ două tipuri de stabilitate.

CORESPONDENȚE ÎNTRE VARIETĂȚI RIGLATE PRIN VARIETĂȚI TANGENTE SUPRAPUSE ÎN S_4 AFIN

DE

ȘTEFANIA RUSCIOR și AGNIA OJOG

În această lucrare se studiază o corespondență între două familii de suprafețe $\{R\}$ și $\{R^*\}$ extinse în sens M a y e r [1] ...[3] astfel încât în punctele corespunzătoare planele tangente să fie semiparalele și așezate în hiperplane tangente suprapuse.

Să considerăm într-un spațiu afin S_4 două varietăți biriglate [4] V și V^* date de ecuațiile :

$$(1) \quad \begin{aligned} \bar{r} &= \bar{\rho}(u) + v\bar{a}(u) + w\bar{b}(u, v), \\ \bar{r}^* &= \bar{\rho}^*(u) + v\bar{a}^*(u) + w^*\bar{b}^*(u, v). \end{aligned}$$

Aceste două varietăți admit ca suprafețe directoare suprafețele riglate (S) și (S^*) date de ecuațiile :

$$\bar{r} = \bar{\rho}(u) + v\bar{a}(u), \quad \bar{r}^* = \bar{\rho}^*(u) + v\bar{a}^*(u)$$

și au generatoarele g și g^* determinate în fiecare punct de versorii $\bar{b}(uv)$ și $\bar{b}^*(uv)$. Există astfel stabilită o corespondență între generatoarele varietății V și generatoarele varietății V^* , determinată de alegerea parametrilor pe varietate. Astfel $u = \text{const.}$ și $v = \text{const.}$ determină două generatoare g și g^* . De-a lungul acestor generatoare construim familia de suprafețe $\{R\}$, respectiv $\{R^*\}$, determinate de $w = \text{const.}$ și $w^* = \text{const.}$

Coordonata proiectivă w ce caracterizează un punct de pe g , caracterizează și o suprafață R din familia $\{R\}$; analog w^* determină o suprafață R^* din familia $\{R^*\}$. Există astfel determinată și o corespondență biunivocă între punctele generatoarelor g și g^* și suprafețele R și R^* considerate.

Planul tangent T dus la o suprafață R într-un punct este determinat de vectorii $\bar{\rho}' + v\bar{a}' + w\bar{b}_u$ și $\bar{a} + w\bar{b}_v$ și este așezat în hiperplanul tangent H dus varietății V în punctul M . Analog planul tangent T^* dus unei suprafețe R^* într-un punct $M^* \in g^*$ este determinat de vectorii $\bar{\rho}^{*'} + v\bar{a}^{*'} + w\bar{b}_u^*$ și $\bar{a}^* + w\bar{b}_v^*$ și este așezat în hiperplanul tangent H^* dus varietății V^* în punctul M^* . Ne propunem să găsim o corespondență între suprafețele R și R^* prin plane tangente T și T^* semiparalelele, în ipoteza că hiperplanele tangente H și H^* care conțin respectiv pe T și T^* sînt suprapuse.

Știm că hiperplanul tangent H este determinat în fiecare punct $M \in g$ de vectorii $\bar{r}_u, \bar{r}_v, \bar{r}_w$ și hiperplanul tangent H^* este determinat de vectorii $\bar{r}_u^*, \bar{r}_v^*, \bar{r}_w^*$. Impunem condiția ca planele tangente T și T^* să fie semiparalele [5]. Rezultă :

$$(2) \quad |\bar{\rho}' + v\bar{a}' + w\bar{b}_u, \bar{a} + w\bar{b}_v, \bar{\rho}^{*'} + v\bar{a}^{*'} + w\bar{b}_u^*, \bar{a}^* + w\bar{b}_v^*| = 0.$$

Ordonînd expresia (2) după coordonatele proiective w și w^* , găsim o relație algebrică de gradul 4 de forma

$$(3) \quad \begin{aligned} & A_{22}w^2w^{*2} + A_{12}ww^{*2} + A_{21}w^2w^* + A_{11}ww^* + \\ & + A_{20}w^2 + A_{02}w^{*2} + A_{10}w + A_{01}w^* + A_{00} = 0. \end{aligned}$$

Coefficienții A_{ik} sînt determinați în funcție de elementele varietăților V și V^* și au expresiile :

$$A_{22} = |\bar{b}_u \bar{b}_v \bar{b}_u^* \bar{b}_v^*|,$$

$$A_{12} = |\bar{\rho}' \bar{b}_u \bar{b}_u^* \bar{b}_v^*| + |\bar{b}_u \bar{a} \bar{b}_u^* \bar{b}_v^*| + v |\bar{a}' \bar{b}_v \bar{b}_u^* \bar{b}_v^*|,$$

$$A_{21} = |\bar{b}_u \bar{b}_v \bar{\rho}^{*'} \bar{b}_v^*| + |\bar{b}_u \bar{b}_v \bar{b}_u^* \bar{a}^*| + v |\bar{b}_u \bar{b}_v \bar{a}^{*'} \bar{b}_v^*|,$$

$$A_{11} = |\bar{\rho}' \bar{b}_v \bar{\rho}^{*'} \bar{b}_v^*| + |\bar{\rho}' \bar{b}_v \bar{b}_u^* \bar{a}^*| + |\bar{b}_u \bar{a} \bar{\rho}^{*'} \bar{b}_v^*| +$$

$$\begin{aligned} & + |\bar{b}_u \bar{a} \bar{b}_u^* \bar{a}^*| + v |\bar{\rho}' \bar{b}_v \bar{a}^{*'} \bar{b}_v^*| + v |\bar{a}' \bar{b}_v \bar{\rho}^{*'} \bar{b}_v^*| + v |\bar{a}' \bar{b}_u \bar{b}_u^* \bar{a}^*| + \\ & + v |\bar{b}_u \bar{a} \bar{a}^{*'} \bar{b}_v^*| + v^2 |\bar{a}' \bar{b}_v \bar{a}^{*'} \bar{b}_v^*|, \end{aligned}$$

$$A_{20} = |\bar{b}_u \bar{b}_v \bar{\rho}^{*'} \bar{a}^*| + v |\bar{b}_u \bar{b}_v \bar{a}^{*'} \bar{a}^*|,$$

$$A_{02} = |\bar{\rho}' \bar{a} \bar{b}_u^* \bar{b}_v^*| + v |\bar{a}' \bar{a} \bar{b}_u^* \bar{b}_v^*|,$$

$$A_{10} = |\bar{\rho}' \bar{b}_v \bar{\rho}^{*'} \bar{a}^*| + |\bar{b}_u \bar{a} \bar{\rho}^{*'} \bar{a}^*| + v |\bar{\rho}' \bar{b}_v \bar{a}^{*'} \bar{a}^*| +$$

$$+ v |\bar{a}' \bar{b}_v \bar{\rho}^{*'} \bar{a}^*| + v |\bar{b}_u \bar{a} \bar{a}^{*'} \bar{a}^*| + v^2 |\bar{a}' \bar{b}_v \bar{a}^{*'} \bar{a}^*|,$$

$$\begin{aligned}
A_{01} &= |\bar{\rho}' \bar{a} \bar{\rho}' \bar{b}_v^*| + |\bar{\rho}' \bar{a} \bar{b}_u^* \bar{a}^*| + v |\bar{\rho}' \bar{a} \bar{a}^* \bar{b}_v^*| + \\
&+ v |\bar{a}' \bar{a} \bar{\rho}' \bar{b}_v^*| + v |\bar{a}' \bar{a} \bar{b}_u^* \bar{a}^*| + v^2 |\bar{a}' \bar{a} \bar{a}^* \bar{b}_v^*|, \\
A_{00} &= |\bar{\rho}' \bar{a} \bar{\rho}' \bar{a}^*| + v |\bar{\rho}' \bar{a} \bar{a}^* \bar{a}^*| + v |\bar{a}' \bar{a} \bar{\rho}' \bar{a}^*| + v^2 |\bar{a}' \bar{a} \bar{a}^* \bar{a}^*|.
\end{aligned}$$

Relația (3) arată că corespondența prin plane tangente semiparalele stabilită între familia de suprafețe $\{R\}$ și $\{R^*\}$ determină între punctele generatoarelor corespunzătoare o corespondență cremoniană de gradul 4.

Această corespondență este stabilită în cazul general când varietățile V și V^* sînt de tip V_0^2 și hiperplanele tangente H și H^* sînt hiperplane distincte. Să studiem corepondența în cazul cînd hiperplanele tangente H și H^* se suprapun. Suprapunerea celor două hiperplane tangente se realizează mai simplu pe varietăți riglate de tip V_1^2 [6], [7], adică în cazul în care hiperplanele tangente H duse de-a lungul generatoarei g formează un fascicul de hiperplane cu baza într-un plan P numit *plan de bază* al generatoarei g . Analog H^* formează un fascicul de hiperplane ce trec prin P^* , planul de bază al generatoarei g^* . În acest caz hiperplanul H dus într-un punct $M \in g$, conține planul P și planul tangent T dus în M suprafeței R . Prin urmare H conține și direcția d comună a planelor T și T^* determinate prin condiția de semiparalelism impusă acestor plane. Hiperplanul H^* dus într-un punct $M^* \in g^*$ conține planul P^* și planul T^* , prin urmare conține și direcția d .

Rezultă că dacă cele două plane de bază P și P^* coincid, hiperplanele tangente H și H^* se suprapun în punctele corespunzătoare determinate de relația (3). În planul de bază P este conținută generatoarea g și dreapta ei caracteristică [6]. Dacă varietatea V^* are ca generatoare g^* , dreapta caracteristică a generatoarei g , planul de bază P este determinat de g și g^* și coincide cu P^* iar hiperplanele tangente H și H^* se suprapun. Impunînd condițiile analitice, pentru ca V și V^* să fie de tip V_1^2 și ca g^* să fie dreapta caracteristică a lui g , rezultă că matricea

$$M = (\bar{a} \bar{a}^* \bar{b}_u \bar{b}_v \bar{b}_u^* \bar{b}_v^* \bar{\rho}' \bar{\rho}')$$

are rangul 3. În acest caz o parte din coeficienții din relația (3) se anulează și anume

$$A_{22} = 0, \quad A_{21} = 0, \quad A_{20} = 0, \quad A_{12} = 0, \quad A_{02} = 0.$$

Obținem astfel

$$(4) \quad A_{11} w w^* + A_{10} w + A_{01} w^* + A_{00} = 0,$$

unde

$$\begin{aligned}
A_{11} &= |\bar{a}' \bar{b}_v \bar{a}^* \bar{b}_v^*|, & A_{10} &= |\bar{a}' \bar{b}_v \bar{a}^* \bar{a}^*|, \\
A_{01} &= |\bar{a}' \bar{a} \bar{a}^* \bar{b}_v^*|, & A_{00} &= |\bar{a}' \bar{a} \bar{a}^* \bar{a}^*|.
\end{aligned}$$

Relația (4) arată că în condițiile problemei corespondența menționată induce între punctele generatoarelor corespunzătoare o omografie. Avem prin urmare următorul rezultat :

Corespondența prin plane tangente semiparalele conținute în același hiperplan tangent, stabilită între două familii de suprafețe $\{R\}$ așezate pe varietăți riglate de tip V_1^2 , determină între punctele generatoarelor corespunzătoare o omografie dată de (4).

Pentru a pune în evidență punctele duble ale omografiei stabilite vom raporta cele două spații liniare H și H^* la același sistem de coordonate. Fie sistemul local considerat dat de versorii fundamentali $\bar{a}(1, 0, 0, 0)$, $\bar{a}'(0, 1, 0, 0)$, $\bar{a}^*(0, 0, 1, 0)$, $\bar{a}''(0, 0, 0, 1)$. În acest sistem vectorii $\bar{b}_v$ și $\bar{b}_v^*$ au componentele α_i și β_i ($i = 1, 2, 3, 4$).

Omografia (4) se exprimă prin ecuația

$$(5) \quad (\alpha_1\beta_3 - \beta_1\alpha_3)w w^* + \alpha_1 w + \beta_3 w^* + 1 = 0.$$

Se constată că punctele duble ale omografiei sînt date de relația

$$(\alpha_1\beta_3 - \beta_1\alpha_3)w^2 + (\alpha_1 + \beta_3)w + 1 = 0.$$

Ele sînt distincte, prin urmare omografia (5) este generală.

Omografia (5) devine parabolică pe varietăți particulare cînd este îndeplinită condiția :

$$(\alpha_1 - \beta_3)^2 + 4\beta_1\alpha_3 = 0.$$

În acest caz punctul dublu al omografiei este

$$w_0 = -\frac{\alpha_1 + \beta_3}{2(\alpha_1\beta_3 - \beta_1\alpha_3)}.$$

El descrie o suprafață S_0 , atunci cînd generatoarea g generează varietatea V .

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CORESPONDANCES ENTRE VARIÉTÉS RÉGLÉES PAR DES
VARIÉTÉS SUPERPOSÉES EN S_4

(Résumé)

On étudie la correspondance par plans tangents semiparallèles appartenant au même hyperplan tangent, établie entre deux familles de surfaces $\{R\}$ situées sur des variétés réglées du type V_1^2 . On trouve que cette correspondance détermine un homographie générale, qui peut devenir parabolique dans un cas particulier.

ON THE CONVOLUTION OF DISTRIBUTIONS (II)

BY

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1. Introduction

In [8] we have defined the convolution of the distributions in the sense of Mikusinski's definition of distributions ([1], [4]) and we finally established a theorem in which was proved the equivalence of our results with the ones presented in [2]. In the following we continue with other considerations on the convolution of distributions.

2. Convolution Equations

In this paper we don't intent to present all aspects connected with this problem. We will expose only some ideas which are to be developed in the author doctoral dissertation [9]. These ideas appear as being very natural consequences of the general results developed in [10], [11] and as well as in the first part of this paper [8]. The established results allow, in particular, to substitute the resolution of a convolution equation in terms of distributions with a convolution equation in terms of indefinitely differentiable functions.

Let us consider the convolution equation

$$(1) \quad h(x) \times z(x) = f(x),$$

where $f(x) = [f_n(x)]$, $h(x) = [h_n(x)]$ are two given distributions and $z(x) = [z_n(x)]$ is an unknown distribution. To solve (1), it means to find a fundamental sequence $\{z_n(x)\}$ such that $\{h_n(x) \times z_n(x)\} \sim \{f_n(x)\}$.

Obviously, we have more such sequences and it is not necessary that all these sequences belong to the same equivalence class because, in general, the uniqueness of the solution $z(x)$ is not assured.

We will make now two important observations, namely:

1° if $\{z_n(x)\}$ is a fundamental sequence such that $\{h_n(x) \times z_n(x)\} \sim \{f_n(x)\}$, then there exists a fundamental sequence $\{f'_n(x)\} \sim \{f_n(x)\}$ such that $h_n(x) \times z_n(x) = f'_n(x)$.

Indeed, if we let $\{f'_n(x)\}$ be the sequence $\{h_n(x) \times z_n(x)\}$ the assertion follows immediately.

2° if $\{z_n(x)\}$ is verifying the equation $h_n(x) \times z_n(x) = f_n(x)$, then it results that the distribution $z(x) = [z_n(x)]$ is a solution of (1) and we can conclude:

Theorem 5. *The solving of equation (1) is reduced to determine a fundamental sequence $\{z_n(x)\}$, which terms verifies the equation*

$$(2) \quad h_n(x) \times z_n(x) = f_n(x).$$

Finally, referring to the independence of the solutions with respect to the choice of sequences $\{f_n(x)\}$ and $\{h_n(x)\}$, from the equivalence class of $f(x)$ and respectively of $h(x)$ we have:

Theorem 6. *The set of the solutions of equation (1) is independent with respect to the choice of the sequences $\{f_n(x)\}$ and $\{h_n(x)\}$.*

Proof. Indeed, if $\{f'_n(x)\} \sim \{f_n(x)\}$, $\{h'_n(x)\} \sim \{h_n(x)\}$ and $z(x) = [z_n(x)]$ is a solution of (1), with $\{z_n(x)\}$ verifying the equation (2), then we will have $\{h'_n(x) \times z_n(x)\} \sim \{f'_n(x)\}$, hence $z(x) = [z_n(x)]$ verifies (1) with respect to $\{f'_n(x)\}$ and $\{h'_n(x)\}$.

Conversely, if $z'(x) = [z'_n(x)]$ is a solution of (1) such as $h'_n(x) \times z'_n(x) = f'_n(x)$, then we will have $\{h_n(x) \times z'_n(x)\} \sim \{f'_n(x)\}$, but obviously we cannot assert nothing about the equivalence of the sequences $\{z_n(x)\}$ and $\{z'_n(x)\}$, excepting the cases when the uniqueness of the solution (1) is assured.

Referring to the problem of existence of a solution for (1), we give here a result which is similar with one obtained by Hörmander [7]. Let us mention that other details are to be found in [9].

Theorem 7. *If equation (2) defines the fundamental sequence $\{z_n(x)\}$ in Ω_2 , for any distribution $f(x) = [f_n(x)] \in \mathcal{M}'(\Omega_1)$ and $h(x) = [h_n(x)] \in \mathcal{M}_0$, then the pair (Ω_1, Ω_2) is $\overset{\vee}{h}$ -strictly convex, and, conversely, if $h(x)$ is reversible and the pair (Ω_1, Ω_2) is $\overset{\vee}{h}$ -strictly convex, then equation (2) defines in Ω_2 the fundamental sequence $\{z_n(x)\}$ for any distribution $f(x) \in \mathcal{M}'(\Omega_1)$.*

3. Applications

We intend to give two applications which reflect some of the advantages of this point of view in the theory of distributions.

1. A first problem which we will consider is the following one: Let us determine the law of motion of an oscillator which is subjected to some impulses μ_{k+1} inversely proportional with $\dot{y}(t_{k+1}-0)$ and applied at the

moments t_{k+1} , when $y(t_{k+1}) = 0$ [5], but let us consider the case when $\mu_{k+1} = a_{k+1} \delta_{t_{k+1}}$.

The equation of motion is

$$(3) \quad y'' + \gamma_1 y' + \gamma_2 y = 0$$

and let us consider the roots of the characteristic equation be $\gamma_{1,2} = \alpha + i\beta$. If t_0 is the initial moment and we impose the initial conditions $y(t_0) = 0$, $y'(t_0) = v_0$ the solution of (3) can be written

$$(4) \quad y(t) = C e^{\alpha t} \sin \beta(t - \tau).$$

Imposing the condition that $y(t_k) = 0$ for $t_k = t_0 + \frac{2k\pi}{\beta}$ we obtain $\tau = t_k - \frac{m\pi}{\beta} = t_0 + \frac{m'\pi}{\beta}$ (see [7]), and hence

$$(5) \quad y(t) = C' e^{\alpha t} \sin \beta(t - t_0).$$

Analogously, from $y'(t_0) = v_0$ it results that $C' = \frac{1}{\beta} v_0$, and, finally

$$(6) \quad y_0(t) = \frac{1}{\beta} v_0 e^{\alpha t} \sin \beta(t - t_0).$$

If we consider now the perturbed equation and take the impulses $\mu_{k+1} = a_{k+1} \delta_{t_{k+1}}$,

$$(7) \quad y'' + \gamma_1 y' + \gamma_2 y = \sum_{k=0}^{\infty} a_{k+1} \delta_{t_{k+1}}$$

then if we consider the distribution $\delta_{t_{k+2}}$ given by the fundamental sequence

$$(8) \quad \delta_{n,k+1}(t) = \begin{cases} \frac{1}{2} n e^{n(t-t_{k+1})} & \text{for } t \leq t_{k+1} \\ \frac{1}{2} n e^{-n(t-t_{k+1})} & \text{for } t \geq t_{k+1} \end{cases}$$

and a neighborhood of t_{k+1} which does not contain the points t_k and t_{k+2} , the equation (7) can be written in this neighborhood

$$(7') \quad y'' + \gamma_1 y' + \gamma_2 y = a_{k+1} \delta_{n,k+1}(t)$$

or, equivalently,

$$(9') \quad y'' + \gamma_1 y' + \gamma_2 y = \frac{1}{2} a_{k+1} n e^{n(t-t_{k+1})} \quad \text{for } t \leq t_{k+1}$$

and

$$(9'') \quad y'' + \gamma_1 y' + \gamma_2 y = \frac{1}{2} a_{k+1} n e^{-n(t-t_{k+1})} \quad \text{for } t \geq t_{k+1}.$$

The solutions of this equations can be written immediately and are

$$(11') \quad y_{n,k+1}^1(t) = y_0(t) + \frac{1}{2} \frac{a_{k+1} n e^{n(t-t_{k+1})}}{n^2 + \gamma_1 n + \gamma_2} \quad \text{for } t \leq t_{k+1}$$

and

$$(11'') \quad y_{n,k+1}^2(t) = y_0(t) + \frac{1}{2} \frac{a_{k+1} n e^{-n(t-t_{k+1})}}{n^2 - \gamma_1 n + \gamma_2} \quad \text{for } t \geq t_{k+1}.$$

If we observe that

$$\delta_{n,k+1}^{(-2)}(t) = \begin{cases} 2n e^{n(t-t_{k+1})} & \text{for } t \leq t_{k+1} \\ t - t_{k+1} + \frac{1}{2n} e^{-n(t-t_{k+1})} & \text{for } t \geq t_{k+1} \end{cases}$$

then

$$[\delta_n^{(-2)}(t)] = \begin{cases} 0 & \text{for } t \leq t_{k+1} \\ t - t_{k+1} & \text{for } t \geq t_{k+1} \end{cases}$$

and

$$(11) \quad \begin{cases} y_{n,k+1}^1(t) = y_0 + \frac{a_{k+1}}{1 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2}} \delta_{n,k+1}^{(-2)}(t) & \text{for } t \leq t_{k+1} \\ y_{n,k+1}^2(t) = y_0(t) + \frac{a_{k+1}}{1 - \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2}} (\delta_{n,k+1}^{(-2)}(t) - t + t_{k+1}) & \text{for } t \geq t_{k+1} \end{cases}$$

If we denote $\delta_{k+1}^{(-2)}(t) = [\delta_{n,k+1}^{(-2)}(t)]$, it results the solution

$$(12) \quad y_{k+1}(t) = a_{k+1} [\delta_{k+1}^{(-2)}(t) - H_{k+1}(t)(t - t_{k+1})] + y_0(t).$$

Finally,

$$y(t) = y_0(t) + \sum_{k=0}^{\infty} a_{k+1} [\delta_{k+1}^{(-2)}(t) - H_{k+1}(t)(t - t_{k+1})],$$

where $H_{k+1}(t)$ is the Heaviside function.

In the following we give a physical signification of this point of view, concerning the solution of this problem, namely, it is known that the impulses are not ideal and if we consider the interval $(t_{k+1} - \frac{1}{n}, t_{k+1} + \frac{1}{n})$ as being the period of action of μ_{k+1} , then if in this interval μ_{k+1} is given by the expression $\mu_{k+1} = a_{k+1} \delta_{t_{k+1}}(t)$, and $\delta_{t_{k+1}}(t) = [\delta_{n,k+1}(t)]$, let be N_0 such that $\text{supp } \delta_{n,k+1}(t) \subset (t_{k+1} - \frac{1}{n}, t_{k+1} + \frac{1}{n})$ for all $n \geq N_0$.

With this conditions we can assimilate the motion of the oscillator working under the ideal impulse μ_{k+1} applied to the moments t_{k+1} with the motion of an oscillator working under the continuous action of $\mu_{n,k+1}(t) = a_{k+1} \delta_{n,k+1}(t)$ with $n > N$.

2. The problem on which we are concerned in the following have been considered by P. P. Teodorescu and W. Kees in [6], from the Schwartz's point of view in theory of distributions.

Let us consider now the free transversal vibrations of a infinite bar, which are described by the differential equation [6]

$$(13) \quad \frac{\partial^4}{\partial x^4} v(x, t) + \frac{1}{C^2} \frac{\partial^2}{\partial t^2} v(x, t) = 0$$

(the signification of C can be found in [6]).

We want to solve this equation with the initial conditions

$$(14) \quad v(x, 0) = f_1(x), \quad \frac{\partial}{\partial t} v(x, 0) = C \frac{d^2 f_2(x)}{dx^2},$$

where $f_1(x)$ and $f_2(x)$ are two given distributions.

We resort to the solution of equation

$$(15) \quad \frac{\partial^4}{\partial x^4} v_n(x, t) + \frac{1}{C^2} \frac{\partial^2}{\partial t^2} v_n(x, t) = 0,$$

with the initial conditions

$$(16) \quad v_n(x, 0) = f_{1n}(x), \quad \frac{\partial}{\partial t} v_n(x, 0) = C \frac{d^2 f_{2n}(x)}{dx^2},$$

where $f_1(x) = [f_{1n}(x)]$, $f_2(x) = [f_{2n}(x)]$ and $v(x, t)$ will be the distribution given by the fundamental sequence $\{v_n(x, t)\}$, because this equation can be considered as a convolution equation and we can apply Theorem 7.

If we suppose in addition that the distributions $f_1(x)$ and $f_2(x) \in \mathcal{M}'_{\mathfrak{F}}$ [10], and we look for a solution $v(x, t)$ in $\mathcal{M}'_{\mathfrak{F}}$ with respect to the variable x , then, if we consider the sequences $f_{1n}(x)$ and $f_{2n}(x)$ as being $\mathfrak{F}$ -admissible [10], and, if we denote the Fourier transform of $v_n(x, t)$ by

$$\hat{v}_n(y, t) = \frac{1}{2\pi} \int_R v_n(x, t) e^{-ixy} dx$$

then, after applying the Fourier transform to equation (15), we obtain

$$(17) \quad y^4 \hat{v}_n(y, t) + \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \hat{v}_n(y, t) = 0,$$

with the initial conditions

$$(18) \quad \hat{v}_n(y, 0) = \hat{f}_{1n}(y), \quad \frac{\partial}{\partial t} \hat{v}_n(y, 0) = -y^2 C \hat{f}_{2n}(y).$$

By (17) it results

$$(19) \quad \hat{v}_n(y, t) = A_n(y) \cos Cy^2t + B_n(y) \sin Cy^2t$$

and the initial conditions (18) give

$$(20) \quad A_n(y) = \hat{f}_{1n}(y) \quad \text{and} \quad B_n(y) = -\hat{f}_{2n}(y).$$

Hence

$$(21) \quad \hat{v}_n(y, t) = \hat{f}_{1n}(y) \cos Cy^2t - \hat{f}_{2n}(y) \sin Cy^2t.$$

If we are going to the inverse Fourier transform and remember that

$$(\mathcal{F}^{-1} \cos Cy^2t) = \frac{1}{2\sqrt{2\pi Ct}} \left(\cos \frac{x^2}{4Ct} + \sin \frac{x^2}{4Ct} \right),$$

$$(\mathcal{F}^{-1} \sin Cy^2t) = \frac{1}{2\sqrt{2\pi Ct}} \left(\cos \frac{x^2}{4Ct} - \sin \frac{x^2}{4Ct} \right),$$

then we find for the $v_n(x, t)$ the expression

$$(22) \quad v_n(x, t) = \frac{1}{2\sqrt{2\pi Ct}} \left[[f_{1n}(x) \times \left(\cos \frac{x^2}{4Ct} + \sin \frac{x^2}{4Ct} \right) - f_{2n}(x) \times \left(\cos \frac{x^2}{4Ct} - \sin \frac{x^2}{4Ct} \right)] \right].$$

Finally, as in [6],

$$(23) \quad v(x, t) = [v_n(x, t)] = \frac{1}{2\sqrt{2\pi Ct}} \left[f_1(x) \times \left(\cos \frac{x^2}{4Ct} + \sin \frac{x^2}{4Ct} \right) - f_2(x) \times \left(\cos \frac{x^2}{4Ct} - \sin \frac{x^2}{4Ct} \right) \right].$$

The two important advantages of this point of view in this problem with respect to that used in [6] consist firstly in the fact that equation (13) can be considered in the set of the distributions, and, consequently, we can disregard all the considerations which has been made in [6] to obtain the equation (8.12.3). Secondly, because the functions $v_n(x, t)$ are indefinitely differentiable, we can consider equation (15) as being an usual differential equation and hence the Laplace transform is not necessary.

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ASUPRA CONVOLUȚIEI DISTRIBUȚIILOR (II)

(Rezumat)

Se studiază, în contextul teoriei dezvoltate în [8] și ținând seama de rezultatele prezentate în [10], [11], unele aspecte legate de rezolvarea ecuațiilor de convoluție în mulțimea distribuțiilor [7], aspecte aprofundate în [9]. În final se consideră două aplicații la probleme de mecanică [5], [6].

ASUPRA UNEI CLASE DE ALGEBRE NILPOTENTE DE ORDINUL CINCI

DE

I. ENESCU

1. În cercetarea algebrilor reale asociative, comutative și cu unitate principală, de ordinul șase, intervine clasa de algebre nilpotente de ordinul cinci, dată prin tabela

(1)

	u_1	u_2	u_3	u_4	u_5
u_1	u_3	u_5	pu_5	0	0
u_2	u_5	u_4	0	qu_5	0
u_3	pu_5	0	0	0	0
u_4	0	qu_5	0	0	0
u_5	0	0	0	0	0

unde p și q sînt parametri reali arbitrari.

În această Notă ne propunem determinarea pînă la izomorfism, a tipurilor de algebre (1) corespunzătoare valorilor reale ale parametrilor p și q .

2. Se constată, prin calcul direct, că oricare ar fi p și q reali, orice algebra (1) este asociativă și comutativă.

La fel, se constată că algebra (1) este nilpotentă de indice patru, pentru orice valori p și q care verifică relația $p^2 + q^2 \neq 0$.

În schimb, pentru $p = q = 0$, algebra (1) corespunzătoare, este nilpotentă de indice trei, ceea ce stabilește

Teorema 1. *Cazul $p = q = 0$ conduce la un tip neizomorf de algebra (1) și el este reprezentat prin tabela*

$$(2) \quad \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & u_3 & u_5 & 0 & 0 & 0 \\ u_2 & u_5 & u_4 & 0 & 0 & 0 \\ u_3 & 0 & 0 & 0 & 0 & 0 \\ u_4 & 0 & 0 & 0 & 0 & 0 \\ u_5 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

3. Considerăm deci, numai cazurile pentru care $p^2 + q^2 \neq 0$ și, să schimbăm baza din (1), după relațiile :

$$u_1 = v_1, \quad u_2 = v_2, \quad u_3 = pv_3, \quad u_4 = qv_4, \quad u_5 = v_5.$$

Se obține tabela :

$$(1') \quad \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & pv_3 & v_5 & v_5 & 0 & 0 \\ v_2 & v_5 & qv_4 & 0 & v_5 & 0 \\ v_3 & v_5 & 0 & 0 & 0 & 0 \\ v_4 & 0 & v_5 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

Mai departe, schimbarea bazei din (1') prin

$$v_2 = w_1, \quad v_1 = w_2, \quad v_4 = w_3, \quad v_3 = w_4, \quad v_5 = w_5,$$

conduce la tabela

$$(1'') \quad \begin{array}{c|ccccc} & w_1 & w_2 & w_3 & w_4 & w_5 \\ \hline w_1 & qw_3 & w_5 & w_5 & 0 & 0 \\ w_2 & w_5 & pw_4 & 0 & w_5 & 0 \\ w_3 & w_5 & 0 & 0 & 0 & 0 \\ w_4 & 0 & w_5 & 0 & 0 & 0 \\ w_5 & 0 & 0 & 0 & 0 & 0 \end{array},$$

care dovedește

Teorema 2. Schimbarea între ei, a parametrilor p și q , nu modifică tipul de algebră A .

4. Fie acum $p = 0$ și $q \neq 0$. Atunci, după (1), algebra corespunzătoare are tabela :

	u_1	u_2	u_3	u_4	u_5
u_1	u_3	u_5	0	0	0
u_2	u_5	u_4	0	qu_5	0
u_3	0	0	0	0	0
u_4	0	qu_5	0	0	0
u_5	0	0	0	0	0

Efectuînd aici schimbarea de bază :

$$qu_1 = v_1, \quad u_2 = v_2, \quad q^2u_3 = v_3, \quad u_4 = v_4, \quad qu_5 = v_5,$$

se obține

(3)

	v_1	v_2	v_3	v_4	v_5
v_1	v_3	v_5	0	0	0
v_2	v_5	v_4	0	v_5	0
v_3	0	0	0	0	0
v_4	0	v_5	0	0	0
v_5	0	0	0	0	0

Tabela (3) arată că toate algebrele (1) pentru care $p = 0$ și $q \neq 0$ sînt izomorfe, ele putînd fi reprezentate prin algebra ce corespunde valorii $q = 1$, dată prin (3).

Rezultatul de mai sus, coroborat cu teorema 2, stabilește

Teorema 3. *Toate algebrele (1), corespunzătoare valorii zero a unuia, oricare, din cei doi parametri (celălalt fiind diferit de zero), sînt izomorfe și ele se pot reprezenta prin tabela (3).*

5. Rămîne de cercetat cazul $p^2 + q^2 \neq 0$, $p \neq 0$ și $q \neq 0$.

Fie $p = q \neq 0$. Atunci, schimbarea de bază :

$$\frac{1}{p}u_1 = \varepsilon_1, \quad \frac{1}{p}u_2 = \varepsilon_2, \quad \frac{1}{p^2}u_3 = \varepsilon_3, \quad \frac{1}{p^2}u_4 = \varepsilon_4, \quad \frac{1}{p^2}u_5 = \varepsilon_5,$$

efectuată în (1), conduce, în cazul egalității celor doi parametri, la tabela :

$$(4) \quad \begin{array}{c|ccccc} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ \hline \varepsilon_1 & \varepsilon_3 & \varepsilon_5 & \varepsilon_5 & 0 & 0 \\ \varepsilon_2 & \varepsilon_5 & \varepsilon_4 & 0 & \varepsilon_5 & 0 \\ \varepsilon_3 & \varepsilon_5 & 0 & 0 & 0 & 0 \\ \varepsilon_4 & 0 & \varepsilon_5 & 0 & 0 & 0 \\ \varepsilon_5 & 0 & 0 & 0 & 0 & 0 \end{array},$$

de unde rezultă

Teorema 4. *Toate algebrele (1) corespunzătoare valorilor egale și diferite de zero ale parametrilor, sînt izomorfe și ele se pot reprezenta prin aceea corespunzătoare lui $p = q = 1$, a cărei tabelă de înmulțire este dată de (4).*

6. Considerînd acum cazul $p^2 + q^2 \neq 0$, $p \neq q$, $p \neq 0$, $q \neq 0$, să schimbăm baza algebrei A după relațiile :

$$\frac{1}{p}u_1 = v_1, \quad \frac{1}{q}u_2 = v_2, \quad \frac{1}{p^2}u_3 = v_3, \quad \frac{1}{q^2}u_4 = v_4, \quad \frac{1}{pq}u_5 = v_5.$$

Utilizînd tabela (1) se ajunge, după calcule evidente, la

$$(5) \quad \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & v_3 & v_5 & \frac{1}{\alpha}v_5 & 0 & 0 \\ v_2 & v_5 & v_4 & 0 & \alpha v_5 & 0 \\ v_3 & \frac{1}{\alpha}v_5 & 0 & 0 & 0 & 0 \\ v_4 & 0 & \alpha v_5 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 \end{array},$$

unde $\alpha = p/q$, ceea ce dovedește

Teorema 5. *Algebrele (1) depind numai de raportul p/q .*

Se observă că, în cazul particular $\alpha = 1$, se regăsește tipul dat de (4).

Să schimbăm în (5) baza v_1 după relațiile :

$$\alpha v_1 = w_1, \quad \alpha v_2 = w_2, \quad \alpha^2 v_3 = w_3, \quad \alpha^2 v_4 = w_4, \quad \alpha^2 v_5 = w_5.$$

Se ajunge astfel, la tabela :

$$(5') \quad \begin{array}{c|ccccc} & w_1 & w_2 & w_3 & w_4 & w_5 \\ \hline w_1 & w_3 & w_5 & w_5 & 0 & 0 \\ w_2 & w_5 & w_4 & 0 & \alpha^2 w_5 & 0 \\ w_3 & w_5 & 0 & 0 & 0 & 0 \\ w_4 & 0 & \alpha^2 w_5 & 0 & 0 & 0 \\ w_5 & 0 & 0 & 0 & 0 & 0 \end{array} ,$$

care arată că toate algebrele A , corespunzătoare valorilor $\alpha > 0$, sînt izomorfe cu acelea ce corespund respectiv valorilor $\alpha < 0$.

Să schimbăm în (5) baza, după

$$\alpha v_2 = e_1, \quad \alpha^2 v_4 = e_2, \quad \alpha^4 v_5 = e_3, \\ (\alpha^2 - \sqrt[3]{\alpha^2}) v_4 + \alpha \sqrt[3]{\alpha^2} v_1 = e_4, \quad \alpha^2 \sqrt[3]{\alpha^4} v_3 = e_5.$$

Atunci, calculul produselor $e_i e_j$, efectuate după tabela (5), conduce la tabela

$$(4') \quad \begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline e_1 & e_2 & e_3 & 0 & e_3 & 0 \\ e_2 & e_3 & 0 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 & 0 & 0 \\ e_4 & e_3 & 0 & 0 & e_5 & e_3 \\ e_5 & 0 & 0 & 0 & e_3 & 0 \end{array} .$$

Dacă acum, în (5), schimbăm baza prin

$$\alpha v_2 = \varepsilon_1, \quad v_4 = \varepsilon_2, \quad \alpha^2 v_5 = \varepsilon_3, \quad \alpha v_1 = \varepsilon_4, \quad \alpha^2 v_3 = \varepsilon_5,$$

se obține tabela :

$$(5'') \quad \begin{array}{c|ccccc} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\ \hline \varepsilon_1 & \alpha^2 \varepsilon_2 & \varepsilon_3 & 0 & \varepsilon_3 & 0 \\ \varepsilon_2 & \varepsilon_3 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_4 & \varepsilon_3 & 0 & 0 & \varepsilon_5 & \varepsilon_3 \\ \varepsilon_5 & 0 & 0 & 0 & \varepsilon_3 & 0 \end{array} ,$$

care pune în evidență faptul că algebra (4') corespunde valorii $\alpha = 1$ a parametrului, stabilind astfel

Teorema 6. *Toate algebrele A corespunzătoare valorilor diferite de zero ale parametrului α sînt izomorfe, ele putînd fi reprezentate prin cazul $\alpha = 1$ dat prin tabela (4).*

7. Pentru rezolvarea exhaustivă a problemei, observăm că tipurile date prin (3) și respectiv (4) nu sînt izomorfe, întrucît, pentru (3), divizorii lui zero formează algebra nulă de ordin doi, în timp ce, pentru (4), aceștia formează algebra nulă de ordinul unu.

Există deci numai trei tipuri neizomorfe de algebre A , corespunzătoare cazurilor $p = q = 0$; $p \neq 0$, $q = 0$, ($p = 0$, $q \neq 0$) și $p \neq 0$, $q \neq 0$, reprezentate prin respectiv tabelele (2), (3) și (4).

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SUR UNE CLASSE D'ALGÈBRES NILPOTENTES D'ORDRE CINQ

(Résumé)

On détermine les types non-isomorphes d'algèbres (1), où p et q sont réels arbitraires, en trouvant trois types, donnés par (2), (3), et (4), qui correspondent aux cas $p = q = 0$; $p \neq 0$, $q = 0$ ($p = 0$, $q \neq 0$) et $p \neq 0$, $q \neq 0$.

REMARKS ON SOME RIEMANNIAN SPACES

BY

CĂTĂLINA NIȚESCU

1. L. P. Eisenhart [1] defines the curvature tensor in a Riemannian space as being the tensor with the components given by

$$(1) \quad R_{jkl}^i = \frac{\partial}{\partial x^k} \{ \begin{smallmatrix} i \\ j l \end{smallmatrix} \} - \frac{\partial}{\partial x^l} \{ \begin{smallmatrix} i \\ j k \end{smallmatrix} \} + \{ \begin{smallmatrix} i \\ s k \end{smallmatrix} \} \{ \begin{smallmatrix} s \\ j l \end{smallmatrix} \} - \{ \begin{smallmatrix} i \\ s l \end{smallmatrix} \} \{ \begin{smallmatrix} s \\ j k \end{smallmatrix} \}.$$

If R_{jkl}^i is contracted for i and l we obtain the Ricci tensor:

$$(2) \quad R_{jk} = R_{jki}^i = \frac{\partial}{\partial x^k} \{ \begin{smallmatrix} i \\ j i \end{smallmatrix} \} - \frac{\partial}{\partial x^i} \{ \begin{smallmatrix} i \\ j k \end{smallmatrix} \} + \{ \begin{smallmatrix} i \\ s k \end{smallmatrix} \} \{ \begin{smallmatrix} s \\ j i \end{smallmatrix} \} - \{ \begin{smallmatrix} i \\ s i \end{smallmatrix} \} \{ \begin{smallmatrix} s \\ j k \end{smallmatrix} \}.$$

Making use of the Christoffel'symbols and the expression $\{ \begin{smallmatrix} i \\ j l \end{smallmatrix} \} = \frac{\partial \log \sqrt{g}}{\partial x^l}$, where $g = |g_{ij}|$, the relation (2) becomes:

$$(3) \quad R_{jk} = \frac{\partial^2 \log \sqrt{g}}{\partial x^j \partial x^k} - \frac{\partial}{\partial x^i} \{ \begin{smallmatrix} i \\ j k \end{smallmatrix} \} + \{ \begin{smallmatrix} i \\ j l \end{smallmatrix} \} \{ \begin{smallmatrix} l \\ i k \end{smallmatrix} \} - \{ \begin{smallmatrix} i \\ j k \end{smallmatrix} \} \frac{\partial \log \sqrt{g}}{\partial x^i}.$$

2. We define the space $\bar{V}_n$ as being the Riemannian space for which the Riemannian curvature tensor fulfils the condition:

$$(4) \quad R_{jkl, th}^m = R_{hkl, ij}^m,$$

where $_{,}$ denotes the covariant differentiation with respect to the tensor g_{ij} .

If in (4) we contract for m and l , we obtain:

$$(5) \quad R_{jk, th} = R_{hk, ij}.$$

Multiplying the relation (5) by g^{ki} and taking into account that $g_{,l}^{ki} = 0$, we have:

$$(6) \quad R_{j, th}^i = R_{h, ij}^i.$$

We obtain the

Theorem 1. *Space V_n is the space in which the Ricci tensor satisfies the relation (6).*

We consider the vector $A_i = g^{jk} R_{ij,k}$. We prove the

Theorem 2. *In the space $\tilde{V}_n$, the vector A_i is a gradient.*

Proof. For that the vector A_i be a gradient is necessary to satisfy the condition $\text{rot } A_i = 0$, [1, p. 27], or $A_{i,j} - A_{j,i} = 0$.

In our case :

$$\text{rot } (g^{jk} R_{ij,k}) = \text{rot } (R_{i,k}^k) = R_{i,kj}^k - R_{j,ki}^k.$$

Making use of (6), we have $\text{rot } (g^{jk} R_{ij,k}) = 0$.

3. We define the Riemannian space $\tilde{V}_n$, for which the curvature tensor satisfies the relation :

$$(7) \quad R_{jkl,ih}^m + R_{hkl,ij}^m = 0.$$

If in (7) we contract for m and l , we obtain $R_{jk,ih} + R_{hk,ij} = 0$, which multiplied by g^{ki} , gives

$$(8) \quad R_{j,ih}^i + R_{h,ij}^i = 0.$$

We obtain the

Theorem 3. *The space $\tilde{V}_n$ is a Riemannian one for which Ricci tensor satisfies the relation (8).*

In the space $\tilde{V}_n$ we prove the following

Theorem 4. *In the space $\tilde{V}_n$, the vector A_i is a Killing vector.*

Proof. We recall the conditions so that a vector be a Killing vector :

$$(9) \quad \text{div } A_i = 0, \quad A_{i,j} + A_{j,i} = 0.$$

Multiplying relation (8) by g^{jh} and making use of the symmetry of Ricci tensor, or taking in (8) $j = h$ we have $R_{,ih}^{ih} + R_{,ij}^{ij} = 0$, or

$$(10) \quad R_{,ij}^{ij} = 0.$$

But

$$\text{div } (g^{jk} R_{ij,k}) = \text{div } (R_{i,k}^k), \quad \text{div } (R_{i,k}^k) = g^{kl} R_{i,kl}^k = R_{,kl}^{kl},$$

which in view of (10) is zero.

From (8) and (10) follows the theorem.

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REMARCI ASUPRA UNOR SPAȚII RIEMANNIENE

(Rezumat)

Se definesc două spații Riemanniene particulare și se stabilesc câteva proprietăți ale divergenței tensorului lui Ricci în aceste spații.

ON THE CONTINUITY OF MULTIFUNCTIONS

BY

AL. SABADIȘ

The purpose of this study is to discuss some results concerning the continuity of the multifunctions as well as their illustration with examples and counter examples.

At first we shall give some fundamental notions about multifunctions. By E and F we shall mean two topological spaces (Generally, E and F are not given topology).

Definition 1. A multifunction Γ is a relation of E in F with $\Gamma x \in \mathcal{P}(F)$, $\forall x \in E$.

Definition 2. By the induced function of Γ we mean an usual function $\gamma: E \rightarrow \mathcal{P}(F)$ with $\gamma(x) = \Gamma x$.

From this definition it results that by the topologization of $\mathcal{P}(F)$, the continuity of Γ can be expressed by the continuity of its induced.

Definition 3. By the lower inverse of Γ we mean a multifunction $\Gamma^-: F^* \rightarrow E$ with $\Gamma^-y = \{x \mid x \in E, y \in \Gamma x\}$ (We use F^* to denote the set of points $y \in F$ for which there exists $x \in E$ with $y \in \Gamma x$).

We shall put $\Gamma^-B = \{x \mid x \in E, \Gamma x \cap B \neq \emptyset\}$, $\forall B \subset F^*$.

Definition 4. By the upper inverse of Γ we mean application $\Gamma^+: \mathcal{P}(F^*) \rightarrow E$ with $\Gamma^+B = \{x \mid x \in E, \Gamma x \subset B\}$, $\forall B \in \mathcal{P}(F^*)$.

Now we can introduce the notion of continuous multifunction in a point.

Definition 5. Γ is lower semicontinuous in point $x_0 \in E$ if for $\forall D \in \tau_F$ with $D \cap \Gamma x_0 \neq \emptyset$ there exists a neighbourhood of x_0 , $V_{x_0}(D) \subset E$, so that for $\forall x \in V_{x_0}(D)$ we have $\Gamma x \cap D \neq \emptyset$.

Definition 6. Γ is upper semicontinuous in point $x_0 \in E$ if for $\forall D \in \tau_F$ with $\Gamma x_0 \subset D$ there exists a neighbourhood $V_{x_0}(D)$ so that $\forall x \in V_{x_0}(D)$, $\Gamma x \subset D$.

Definition 7. Γ is continuous in x_0 if it is upper and lower semicontinuous in this point. Γ will be continuous on a set if it is continuous in any of its points.

Further one we shall give some conditions equivalent tot hese defini-tions making a comparison with the results known for usual functions.

Theorem 1. *The following statements are equivalent :*

- a) $\Gamma : E \rightarrow F$ is upper semicontinuous on E ;
- b) $\forall A \subset F, A = \bar{A} \Rightarrow \Gamma^{-}A$ is closed in E ;
- c) $\forall D \in \tau_F \Rightarrow \Gamma^{+}D \in \tau_E$.

Proof. We shall prove that (a) $\Leftrightarrow$ (c). Indeed, we assume that Γ is upper semicontinuous on E and $\Gamma^{+}D \neq \emptyset$. By definition $\forall x_0 \in \Gamma^{+}D \Rightarrow \Gamma x_0 \subset D$. We shall show that $x_0 \in \text{Int}(\Gamma^{+}D)$. From $\Gamma x_0 \subset D$ in the above hypothesis, it results the existence of a neighbourhood $V_{x_0}(D)$ so that $\forall x \in V_{x_0}(D), \Gamma x \subset D$ i. e. $\forall x \in V_{x_0}(D) \Rightarrow x \in \Gamma^{+}D$. Therefore $V_{x_0}(D) \in \Gamma^{+}D$ which shows that $\Gamma^{+}D \in \tau_E$.

Reciprocally we assume that $\forall D \in \tau_F, \Gamma^{+}D \in \tau_E$. Let be $x_0 \in E$ and $D \in \tau_F$ with $\Gamma x_0 \subset D$. By definition we have $\Gamma^{+}D = \{x \mid x \in E, \Gamma x \subset D\}$. Therefore $x_0 \in \Gamma^{+}D$ which is just the neighbourhood $V_{x_0}(D)$ from definition 6.

Now we prove that (a) $\Leftrightarrow$ (b). We first assume that Γ is upper semicontinuous on E . Be $A \subset F, A = \bar{A}$ and $\forall x \in E$ so that $\Gamma x \cap A = \emptyset$ i. e. $x \in \Gamma^{-}A$. We denote $B = F - A$. There exists an open D with $\Gamma D \subset B$ and $x \in D$. Then $\Gamma D \cap A = \emptyset$ i. e. $x \in D \subset E - \Gamma^{-}A$ which shows that $E - \Gamma^{-}A \in \tau_E$. It results that $\Gamma^{-}A$ is open in E . We assume now that $\forall A \subset F, A = \bar{A}$ is closed in E . Let be $x \in E$, and $\forall D \in \tau_F$ with $\Gamma x \subset D$. According to the hypothesis $\Gamma^{-}(F - D)$ is closed. But $\Gamma x \cap (F - D) = \emptyset \Rightarrow x \in \Gamma^{-}(F - D)$. Therefore $x \in E - \Gamma^{-}(F - D) \in \tau_E$. As $\forall x' \in E - \Gamma^{-}(F - D), \Gamma x' \cap (F - D) = \emptyset \Rightarrow \Gamma x' \subset D$, which proves that Γ is upper semicontinuous.

In the end (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (b) $\Rightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). Q. e. d.

Theorem 2. *The following statements are equivalent :*

- a) $\Gamma : E \rightarrow F$ is lower semicontinuous on E ;
- b) $\forall A \subset F, A \in \tau_F \Rightarrow \Gamma^{-}A \in \tau_E$.

Proof. (a) $\Leftrightarrow$ (b). We assume that Γ is lower semicontinuous on E , i. e. $\forall x_0 \in E$ and $\forall D \in \tau_F$ with $D \cap \Gamma x_0 \neq \emptyset$; there exists a neighbourhood $V_{x_0}(D) \subset E$, so that for $\forall x \in V_{x_0}(D)$ we have $\Gamma x \cap D \neq \emptyset$. Let be $A \in \tau_F$ with $\Gamma^{-}A \neq \emptyset$. We shall show that $\Gamma^{-}A$ is an open set. Indeed, $\forall x \in \Gamma^{-}A$ we have $\Gamma x \cap A \neq \emptyset$.

According to the hypothesis, it results that there exists a neighbourhood $V_x(A)$ of x so that $\forall y \in V_x(A)$ we have $\Gamma y \cap A \neq \emptyset$, i. e. $V_x(A) \subset \Gamma^{-}A$ which shows that $\Gamma^{-}A$ has only interior points.

(b) $\Rightarrow$ (a). We assume that $\Gamma^{-}A \in \tau_E, \forall A \in \tau_F$. Be $x_0 \in E$ with $\Gamma x_0 \cap A \neq \emptyset$. Then $\forall y \in \Gamma^{-}A$ we have $\Gamma y \cap A \neq \emptyset$. Putting $V_x(D) = \Gamma^{-}A \in \tau_E$ it results (a).

We can see that for usual functions the notion of upper semicontinuous function is equivalent to that of upper semicontinuous function. We shall give examples from which we can see that, in the case of multifunctions, the two notions are not equivalent.

Example 1. Let be the interval $[0, 1]$ usually topologized. We define the application $\Gamma: [0, 1] \rightarrow [0, 1]$ by

$$\Gamma x = \begin{cases} \left\{ \frac{x}{2} \right\} & \text{for } x \in \left[0, \frac{1}{2} \right], \\ \left[\frac{1}{4}, \frac{3}{4} \right] & \text{for } x = \frac{1}{2}, \\ \left\{ \frac{x+1}{2} \right\} & \text{for } x \in \left[\frac{1}{2}, 1 \right]. \end{cases}$$

We observe that $\Gamma([0, 1]) = [0, 1]$ i. e. Γ is a surjective application. Be $A \subset [0, 1]$, $A = \bar{A}$ so that $A \subset \left[0, \frac{1}{2} \right]$ or $A \subset \left[\frac{3}{4}, 1 \right]$. In this case

$$\Gamma^{-}A = \begin{cases} f_1^{-1}(A) & \text{for } A \subset \left[0, \frac{1}{4} \right], \\ f_2^{-1}(A) & \text{for } A \subset \left[\frac{3}{4}, 1 \right], \end{cases}$$

where $f_1(x) = \frac{x}{2}$ and $f_2(x) = \frac{x+1}{2}$. These last functions being usual continuous functions, it results that $\Gamma^{-}A$ is closed. When $\frac{1}{4} \in A$ we shall

write $A = A_1 \cup A_2$ where $A_1 \subset \left[0, \frac{1}{4} \right]$ and $\frac{1}{4} \in A_1$, $A_1 = \bar{A}_1$ and $A_2 \subset \left[\frac{1}{4}, 1 \right]$, $A_2 = \bar{A}_2$, $\frac{1}{4} \in A_2$. We shall have $\Gamma^{-}A = \Gamma^{-}A_1 \cup \Gamma^{-}A_2 = f_1^{-1}(A_1) \cup \cup f_2^{-1}(A_2)$ which is a closed set. Therefore, Γ is upper semicontinuous

according to Theorem 1(b). The same will be done when $\frac{3}{4} \in A$, $A = \bar{A}$.

But if $B \subset \left[\frac{1}{4}, \frac{3}{4} \right]$, B being open, then $\Gamma^{-}B = \left\{ \frac{1}{2} \right\}$ which is a closed set.

Therefore Γ does not satisfy Theorem 2(b) and consequently it is not lower semicontinuous.

Example 2. Let be $\Gamma: [0, 1] \rightarrow [0, 1]$ defined by:

$$\Gamma x = \begin{cases} [x, 1] & \text{for } x \in (0, 1), \\ \{1\} & \text{for } x = 0. \end{cases}$$

We observe that any open interval $(a, b) \subset [0, 1]$ has the counterimage $\Gamma^{-}((a, b)) = (0, b)$. And for $0 < a < 1$, $\Gamma^{-}(0, a] = (0, a]$. Consequently, this application is lower semicontinuous (T_2, b) but it is not upper semicontinuous (T_1, b) .

Example 3. Let be $\Gamma_1: R \rightarrow R$ defined by $\Gamma_1 x = (x, +\infty)$. Here $\Gamma_1^-([a, b]) = \{x \mid (x, +\infty) \cap [a, b] \neq \emptyset\} = (-\infty, b)$ i. e. Γ_1 is not upper semicontinuous (T_1, b) . But $\Gamma_1^-((a, b)) = (-\infty, b)$ i. e. Γ_1 is lower semicontinuous (T_2, b) . We prove that $\mathcal{J} = \{\emptyset, R_1; \Gamma_1 x, x \in R\}$ forms a topology on R .

Indeed if $x_1, x_2 \in R \Rightarrow \Gamma_1 x_1, \Gamma_1 x_2 \in \mathcal{J}$ and $\Gamma_1 x_1 \cap \Gamma_1 x_2 = \Gamma_1 z, z \in R, z = \max\{x_1, x_2\}$. And $\bigcup_I \Gamma_1 x_i = \Gamma_1(\bigcup_I x_i)$ for any index family I . Indeed if $\mathcal{J} \in \bigcup_I \Gamma_1 x_i$ then $\exists k \in I$ with $y \in \Gamma_1 x_k \Rightarrow y \in \Gamma_1(\bigcup_I x_i)$. And $z \in \Gamma_1(\bigcup_I x_i) \Rightarrow \exists l \in I$ with $x_l \in \bigcup_I x_i$ and $z \in \Gamma_1(x_l)$, which implies $z \in \bigcup_I \Gamma_1 x_i$. This topology is usually called the lower topology of the straight line and a usual function $f: R \rightarrow R$ is called lower semicontinuous in a point if it is continuous in that point with regard to this topology.

Otherwise, f is lower semicontinuous if and only if $f^{-1}(\Gamma_1 x) = \{y \mid f(y) \in \Gamma_1 x\}$ is open or $f^{-1}(R - \Gamma_1 x) = \{y \mid f(y) \notin \Gamma_1 x\}$ is closed, i. e. the multifunctions $f^{-1} \circ \Gamma_1$ and $f^{-1} \circ (R - \Gamma_1)$ are respectively punctually open and punctually closed.

Example 4. We consider the multifunction $\Gamma_2: R \rightarrow R$ defined by $\Gamma_2 x = (-\infty, x)$. We have $\Gamma_2^-((a, b)) = (a, +\infty)$. And $\Gamma_2^-([a, b]) = \{x \mid (-\infty, x) \cap [a, b] \neq \emptyset\} = (a, +\infty)$. According to $T_2 b$, this application like that from example 3, is lower semicontinuous, without being upper semicontinuous. It is shown as above that the family $S = \{\emptyset, R, \Gamma_2 x, x \in R\}$ forms a topology on R which is called the lower topology of the straight line. An usual real function f is called upper semicontinuous in a point if it is continuous in that point with regard to the topology S . Or, as above, f will be called upper semicontinuous if and only if the multifunctions $f^{-1} \circ \Gamma_2$ and $f^{-1} \circ (R - \Gamma_2)$ are punctually closed and punctually open, respectively. Then, f will be continuous if and only if $f^{-1} \circ \Gamma_1$ is punctually open and $f^{-1} \circ \Gamma_2$ is punctually closed and similarly for $f^{-1} \circ (R - \Gamma_1)$ and $f^{-1} \circ (R - \Gamma_2)$.

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DESPRE CONTINUITATEA MULTIFUNCȚIILOR

(Rezumat)

Se arată echivalența unor condiții suficiente de semicontinuitate a multifuncțiilor și se dau unele exemple și contraexemple.

FUNDAMENTAL INEQUALITIES FOR POLYVIBRATING OPERATORS

BY

SUNDARAM EASWARAN

*Dedicated to Professor D. Mangeron
 on his 65th birthday*

1. In this Note an analogue of P o i n c a r é's Inequality for Laplace operators is given and a new problem for a simple polyvibrating equation is investigated. It is also shown that polyvibrating operators are positive definite on a specified function space.

Let R denote the rectangle $\{a \leq x \leq b; c \leq y \leq d\}$. Let Γ' be the space of continuous functions $u(x, y)$ defined on R such that $u_x(x, y)$, $u_y(x, y)$ and $u_{xy}(x, y)$ are continuous functions on R . By Γ'' we mean the class of functions $u(x, y)$ such that $u(x, y) \in \Gamma'$ and $u_{xy}(x, y) \in \Gamma''$. Let (u, v) denote the inner product

$$(u, v) = \int_a^b \int_c^d u(x, y) v(x, y) dx dy, \quad u \in \Gamma'', \quad v \in \Gamma''.$$

In this article we will study the solution of the M a n g e r o n equation

$$(1) \quad \frac{\partial^4 u(x, y)}{\partial x^2 \partial y^2} = f(x, y), \quad (x, y) \in R.$$

Under the natural boundary conditions

$$(2) \quad u_{xy}(a, y) = u_{xy}(b, y) = u_{xy}(x, c) = u_{xy}(x, d) = 0.$$

The problem of integrating (1) under the conditions (2) can be replaced by the problem of minimizing the functional

$$(3) \quad F[u] = (u_{xyxy}, u) - 2(u, f) = \iint_R [u_{xy}^2(x, y) - 2u(x, y)f(x, y)] dx dy.$$

However the operator $Lu = u_{xyxy}$ is not positive definite on the subspace M of Γ'' satisfying the boundary conditions (2). Since

$$\iint_R u_{xyxy}(x, y) u(x, y) dx dy = \iint_R u_{xy}^2(x, y) dx dy = 0$$

does not necessarily imply that $u \equiv 0$. We also observe that problem (1), (2) is clearly insoluble for arbitrary $f(x, y)$ but we can easily specify the conditions which $f(x, y)$ must satisfy in order that problem is soluble. First of all integrating (1) under the condition (2) we get

$$(4) \quad \int_a^b u_{xyxy}(x, y) dx = \int_a^b f(x, y) dx = 0,$$

$$(5) \quad \int_c^d u_{xyxy}(x, y) dy = \int_c^d f(x, y) dy = 0,$$

$$(6) \quad \iint_{a,c}^{b,d} u_{xyxy}(x, y) dx dy = \iint_{a,c}^{b,d} f(x, y) dx dy = 0.$$

Since (4) and (5) together imply (6), for the solubility of problem (1) and (2) it is necessary that

$$\int_a^b f(x, y) dx = 0, \quad \int_c^d f(x, y) dy = 0.$$

Further if the problem were soluble then it has infinitely many solutions differing by a function $g(x, y)$ such that

$$\frac{\partial^2 g(x, y)}{\partial x \partial y} = 0.$$

But if we choose a solution $u(x, y)$ which satisfies the conditions of the type

$$(7) \quad \int_a^b u(x, y) dx = 0, \quad \int_c^d u(x, y) dy = 0,$$

then clearly such a solution is unique.

2. We have the following inequalities which are given in the form of theorems.

Theorem 1. *If $u(x, y) \in \Gamma'$ then there exists positive constants $\{a_i\}_{1 \leq i \leq 5}$ such that*

$$(8) \quad \alpha_1 \int_a^b \int_c^d u^2(x, y) dx dy + \alpha_2 \left[\int_a^b \int_c^d u(x, y) dx dy \right]^2 \leq \alpha_3 \int_a^b \int_c^d u_{xy}^2(x, y) dx dy +$$

$$+ \alpha_4 \int_c^d \left[\int_a^b u(x, y) dx \right]^2 dy + \alpha_5 \int_a^b \left[\int_c^d u(x, y) dy \right]^2 dx.$$

Proof. Let (x_1, y_1) and (x_2, y_2) be any two points in the interior of R . Then we have

$$(9) \quad u(x_2, y_2) - u(x_1, x_2) - u(x_2, y_1) + u(x_1, y_1) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial^2 u(x, y)}{\partial x \partial y} dx dy.$$

Squaring both sides of the equation (9) and applying Cauchy's inequality to the right hand side we get

$$(10) \quad u^2(x_2, y_2) + u^2(x_1, y_2) + u^2(x_2, y_1) + u^2(x_1, y_1) + 2u(x_1, y_1)u(x_2, y_2) +$$

$$+ 2u(x_2, y_1)u(x_1, y_2) - 2u(x_2, y_2)u(x_1, y_1) - 2u(x_1, y_1)u(x_1, y_2) -$$

$$- 2u(x_1, y_1)u(x_2, y_1) - 2u(x_1, y_1)u(x_1, y_2) \leq$$

$$\leq (b-a)(d-c) \int_a^b \int_c^d u_{xy}^2(x, y) dx dy.$$

Integrating the left hand side with respect to four variables (x_1, y_1, x_2, y_2) in the four dimensional hypercube determined by

$$\{a \leq x_1 \leq b, a \leq x_2 \leq b; c \leq y_1 \leq d, c \leq y_2 \leq d\},$$

we get

$$\begin{aligned}
 & \int_a^b \int_c^d \int_a^b \int_c^d [u^2(x_2, y_2) + u^2(x_1, y_2) + u^2(x_2, y_1) + u^2(x_1, y_2) + \\
 (11) \quad & + 2u(x_1, y_1)u(x_2, y_2) + 2u(x_2, y_1)u(x_1, y_2) - 2u(x_2, y_2)u(x_2, y_1) - \\
 & - 2u(x_1, y_1)u(x_2, y_1) - 2u(x_1, y_1)u(x_1, y_2)] dx_1 dx_2 dy_1 dy_2 \leq \\
 & \leq (b-a)^3 (d-c)^3 \int_a^b \int_c^d u_{xy}^2(x, y) dx dy,
 \end{aligned}$$

$$\begin{aligned}
 & 4(b-a)(d-c) \int_a^b \int_c^d u^2(x, y) dx dy + 4 \left[\int_a^b \int_c^d u(x, y) dx dy \right]^2 - \\
 (12) \quad & - 4(d-c) \int_c^d \left[\int_a^b u(x, y) dx \right]^2 dy - 4(b-a) \int_a^b \left[\int_c^d u(x, y) dy \right]^2 dx \leq \\
 & \leq (b-a)^3 (d-c)^3 \int_a^b \int_c^d u_{xy}^2(x, y) dx dy,
 \end{aligned}$$

which yields the desired result.

Corollary If $u(x, y) \in \Gamma'$ and is such that

$$u(x, c) = u(x, d) = u(a, y) = u(b, y) = 0,$$

then there exists a positive constant C such that

$$(13) \quad \int_a^b \int_c^d u^2(x, y) dx dy \leq C \int_a^b \int_c^d u_{xyxy}^2(x, y) dx dy.$$

Proof. Replacing $u(x, y)$ by $u_{xy}(x, y)$ in (8) and observing the boundary conditions we get

$$\begin{aligned}
 & 4(b-a)(d-c) \int_a^b \int_c^d u_{xy}^2(x, y) dx dy \leq \\
 (14) \quad & \leq (b-a)^3 (d-c)^3 \int_a^b \int_c^d u_{xyxy}^2(x, y) dx dy.
 \end{aligned}$$

Now since $u(a, y) = u(x, c) = u(a, c) = 0$ we get

$$(15) \quad u(x, y) = \iint_{a \ c}^{x \ y} u_{\xi\eta}(\xi, \eta) d\xi d\eta.$$

Applying Cauchy's inequality to (15) we get

$$(16) \quad u^2(x, y) \leq (x - a)(y - c) \iint_{a \ c}^{b \ d} u_{xy}^2(x, y) dx dy.$$

Integrating (16) we get

$$(17) \quad \iint_{a \ c}^{b \ d} u^2(x, y) dx dy \leq \frac{(b - a)^2 (d - c)^2}{4} \iint_{a \ c}^{b \ d} u_{xy}^2(x, y) dx dy.$$

Hence from (14) and (17) we get

$$\begin{aligned} \iint_{a \ c}^{b \ d} u^2(x, y) dx dy &\leq \frac{(b - a)^2 (d - c)^2}{4} \iint_{a \ c}^{b \ d} u_{xy}^2(x, y) dx dy \leq \\ &\leq \frac{(b - a)^4 (d - c)^4}{16} \iint_{a \ c}^{b \ d} u_{xyxy}^2(x, y) dx dy. \end{aligned}$$

Now let $\tilde{M}_0$ denote the subspace of Γ'' satisfying the conditions

- (i) $u(x, y) \in \Gamma''$,
- (ii) $u_{xy}(a, y) = u_{xy}(b, y) = u_{xy}(x, c) = u_{xy}(x, d) = 0$,
- (iii) $\int_a^b u(x, y) dx = 0, \quad \int_c^d u(x, y) dy = 0$.

Then we have the following

Theorem 2. *The operator $Lu = u_{xyxy}(x, y)$ is positive definite on $\tilde{M}_0$.*

Proof. The inequality (8) by the conditions (iii) reduces

$$\iint_{a \ c}^{b \ d} u^2(x, y) dx dy \leq (b - a)^3 (d - c)^3 \iint_{a \ c}^{b \ d} u_{xy}^2(x, y) dx dy.$$

Thus

$$(Lu, u) = \int_a^b \int_c^d u_{xy}^2(x, y) dx dy \geq \frac{1}{(b-a)^3 (d-c)^3} \int_a^b \int_c^d u^2(x, y) dx dy.$$

Thus the theorem is proved.

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INEGALITĂȚI FUNDAMENTALE PENTRU OPERATORI POLIVIBRANȚI

(Rezumat)

Se stabilește un analog al inegalității lui Poincaré pentru operatori Laplace și se cercetează o nouă problemă pentru o ecuație polivibrantă simplă. De asemenea se arată că operatorii polivibranți sînt pozitiv definiți într-un anumit spațiu funcțional.

ON THE EXISTENCE OF SOLUTIONS OF SOME VOLTERRA
INTEGRAL EQUATIONS IN THE FUNCTIONAL SPACE A_c

BY

GH. BANTAȘ

The main aim of the present Note is to expose a theorem of existence in the functional space A_c ([1]) defined below for the solutions of some Volterra integral equations. The working technique employed will be centred on the method elaborated in [2] and on the fixed point theorem of Schauder - Tychonoff [3, p. 456].

Let $\mathbf{R}_a$ be the semiaxis $[a, \infty)$, $\mathbf{R}^p$ the p -dimensional real Euclidean space and $\mathcal{O}_c$ the locally convex and separable vectorial space of the continuous functions $x: \mathbf{R}_a \rightarrow \mathbf{R}^p$, with the topology generated by the denumerable and sufficient family of seminorms

$$|x|_m = \sup_{a \leq t \leq m} \|x(t)\|,$$

where $m \in \mathbf{N}$ and $\|x(t)\|$ is the Euclidean length of the vector $x(t)$.

In the space $\mathcal{O}_c$ we consider the Banach space $\mathcal{O}$ of the bounded functions on $\mathbf{R}_a$, whose norm is defined by the equality and the Banach

$$|x|_{\mathcal{O}} = \sup_{t \geq a} \|x(t)\|$$

space A_c of the absolutely continuous with summable derivative on $\mathbf{R}_a$, whose norm is defined by the equality

$$|x|_{A_c} = \|x(a)\| + \int_a^{\infty} \|x'(s)\| ds.$$

We shall say that a function is absolutely continuous on $\mathbf{R}_a$ if it is absolutely continuous on every finite interval in $\mathbf{R}_a$.

Concerning the space A_c , the following statements may be proved:

1) $A_c \subset \mathcal{Q}_t \subset \mathcal{Q}$ where $\mathcal{Q}_t$ is the Banach space of the functions $x \in \mathcal{Q}$ with finite limit for $t \rightarrow \infty$, and whose norm is the one in $\mathcal{Q}$.

2) The inclusion under 1) holds in the topological sense too.

Let us now consider the integral operator $T: \mathcal{Q}_c \rightarrow \mathcal{Q}_c$ defined by the equality

$$(Tx)(t) = \int_a^t K(t, s) x(s) ds \quad (t \geq a),$$

where K is a square matrix of order p , continuous on the set

$$\mathcal{T} = \{(t, s) | a \leq s \leq t < \infty\}.$$

Regarding the operator T , it may be proved that it is continuous on $\mathcal{Q}_c$ and that the following proposition holds.

Proposition. *If the following conditions*

$$1) \quad \int_a^\infty \|K(t, t)\| dt = I_1 < \infty,$$

$$2) \quad \int_a^\infty \int_a^t \|K'_t(t, s)\| ds dt = I_2 < \infty$$

and K'_t is continuous on $\mathcal{T}$, are satisfied, then the operator T is continuous on $\mathcal{Q}$, $T\mathcal{Q} \subset A_c$ and,

$$\|Tx\|_{A_c} \leq C\|x\|_{\mathcal{Q}}, \quad (x \in \mathcal{Q}),$$

where $C = I_1 + I_2$ ([1]).

Let us consider an integral equation

$$(E) \quad x = (\mathcal{F}_1 + T\mathcal{F}_2)x,$$

where $\mathcal{F}_1: S \rightarrow A_c$ and $\mathcal{F}_2: S \rightarrow \mathcal{Q}$ are two continuous operators on the closed ball S of radius ρ and center O in the space $\mathcal{Q}$.

Let us denote by S_c the ball of radius ρ and center O in the space A_c .

Theorem. *If the equation (E) satisfies the following conditions:*

1) *The hypotheses of the preceding proposition;*

$$2) \quad \|\mathcal{F}_i x\| \leq r_i \quad \forall x \in S \quad (i = 1, 2);$$

3) $\mathcal{F}_1 S \subset A_c$ and $\mathcal{F}_1 S$ is equicontinuous on every segment in $\mathbf{R}_a$;

$$4) \quad r_1 + Cr_2 \leq \rho;$$

then it admits at least one solution $x \in S_c$.

Proof. To prove this theorem let us appeal to the fixed-point theorem of Schauder - Tychonoff [3, p. 456] in which we take the space $\mathcal{C}_c$ as fundamental functional space X , the set S as convex and closed set $\mathcal{X}$ and the following operator as the operator U :

$$Ux = (\mathcal{F}_1 + T(\mathcal{F}_2))x, \quad (x \in S).$$

To place ourselves in the conditions of this theorem it will suffice to show that U is continuous on S in the topology of $\mathcal{C}_c$, that $US \subset S_c$ and that $\overline{US}$ is a compact set in the topology of the space $\mathcal{C}_c$.

The first statement follows from the continuity of the operator T on $\mathcal{C}_c$ and of the operators $\mathcal{F}_i$ on the ball S .

The second statement is justified thus: Let x be some point in S . Taking into account the definition of operators $\mathcal{F}_1$ and $\mathcal{F}_2$ we get:

$$(1) \quad x \in S \Rightarrow \mathcal{F}_1 x \in A_c, \quad \mathcal{F}_2 x \in \mathcal{C}.$$

From here, from condition 1) and from the preceding proposition it follows that

$$(2) \quad \mathcal{F}_2 x \in \mathcal{C} \Rightarrow (T(\mathcal{F}_2))x \in A_c.$$

From (1), (2) and from the property of linearity of the space A_c it follows the relation

$$Ux = \mathcal{F}_1 x + (T(\mathcal{F}_2))x \in A_c,$$

which proves that

$$(3) \quad US \subset A_c.$$

From the definition of the operator U , the preceding proposition and from conditions (2) and (4) we get:

$$x \in S \Rightarrow \|Ux\|_{A_c} \leq \|\mathcal{F}_1 x\|_{A_c} + \|T(\mathcal{F}_2)x\|_{A_c} \leq r_1 + C \|\mathcal{F}_2 x\|_{\mathcal{C}} \leq r_1 + Cr_2 \leq \rho.$$

This one together with (3) lead us to the conclusion that $US \subset S_c \subset S$.

To prove the third statement we shall apply the compactness criterion of Arzelà - Ascoli [3] in the functional space $\mathcal{C}_c$. According to this criterion we shall have to show that $\overline{US}$ is a family of equally bounded and equicontinuous functions on every fixed segment $[a, b] \subset \mathbb{R}_a$. This will be shown as follows: From the definition of operator U and from condition 2) we derive:

$$\|(Ux)(t)\| \leq \|(\mathcal{F}_1 x)(t)\| + \int_a^t \|K(t, s)\| \|(\mathcal{F}_2 x)(s)\| ds \leq r_1 + r_2 \int_a^t \|K(t, s)\| ds$$

for $\forall t \in [a, b]$ and $\forall x \in S$.

$\|K\|$ being a continuous function on the compact $\mathcal{T}_b = \{(t, s)/a \leq s \leq t \leq b\}$, it follows that it is bounded on this compact. Let M_b be the upper bound of this function on the mentioned compact. We have :

$$\|(Ux)(t)\| \leq r_1 + r_2 M_b, \quad \forall t \in [a, b], \quad \forall x \in S.$$

This proves that $\overline{US}$ is a family of equally bounded functions on every finite interval in $\mathbf{R}_a$.

Calling again on the definition of operator U and on condition 2) we conclude that for $\forall t, \tau \in [a, b]$ and $\forall x \in S$ we have :

$$\begin{aligned} (4) \quad & \| (Ux)(t) - (Ux)(\tau) \| \leq \| (\mathcal{F}_1 x)(t) - (\mathcal{F}_1 x)(\tau) \| + \\ & + \int_a^\tau \| K(t, s) - K(\tau, s) \| \| (\mathcal{F}_2 x)(s) \| ds + \int_\tau^t \| K(t, s) \| \| (\mathcal{F}_2 x)(s) \| ds \leq \\ & \leq \| (\mathcal{F}_1 x)(t) - (\mathcal{F}_1 x)(\tau) \| + r_2 \left[\int_a^\tau \| K(t, s) - K(\tau, s) \| ds + \int_\tau^t \| K(t, s) \| ds \right]. \end{aligned}$$

From the continuity of K on the compact $\mathcal{T}_b$ it results the existence of the upper bound K_b of the function $\|K\|$ on $\mathcal{T}_b$ and the uniform continuity of the function K on the compact above :

$$\sup_{a \leq s \leq b} \| K(t, s) - K(\tau, s) \| < \varepsilon$$

for every $t, \tau \in [a, b]$ with $|t - \tau| < \delta_\varepsilon$.

From here, from (4) and from condition 3) it follows — for $\delta_\varepsilon < \varepsilon$ — that

$$\| (Ux)(t) - (Ux)(\tau) \| < \varepsilon [1 + r_2(b - a) + r_2 K_b]$$

for $\forall t, \tau \in [a, b]$ with $|t - \tau| < \varepsilon$ and $\forall x \in S$.

This proves that $\overline{US}$ is a family of equicontinuous functions on any finite interval in $\mathbf{R}_a$.

Corollary 1. *If the equation (E) satisfies the conditions 1) — 3) of the preceding theorem with $\mathcal{Q}$ instead of S , then it has at least a solution $x \in A_c$.*

Corollary 2. Let us consider the integral equation

$$(E') \quad x(t) = b(t) + \int_a^t K(t, s) f(s, x(s)) ds,$$

where K is a square matrix of order p , continuous on the set $\mathcal{T}$, and x, b , and f are continuous p -vectorial functions on $\mathbf{R}_a, \mathbf{R}_a \times \Sigma$ respectively, Σ

being the closed ball with center in O and radius ρ in the Euclidean space $\mathbf{R}^p$.

If the equation (E') satisfies the conditions :

- 1) The hypotheses of the preceding proposition ;
- 2) $b \in A_c$ and $\|f(t, x)\| \leq r$ for $\forall (t, x) \in \mathbf{R}_a \times \Sigma$;
- 3) $\|h\|_{A_c} + Cr \leq \rho$,

then it admits at least a solution $x \in S_c$.

COROLLARY 3. Let us consider the integral equation

$$(E'') \quad x(t) = \int_a^t k(t, s, x(s)) ds + \int_a^t K(t, s) F(s, x(s)) ds,$$

where K is a square matrix of order p , continuous on the set $\mathcal{T}$, and x, k , and F are continuous p -vectorial functions on $\mathbf{R}_a, \mathcal{T} \times \Sigma$ and $\mathbf{R}_a \times \Sigma$ respectively. We shall also assume that the function k'_t exists and is continuous on $\mathcal{T} \times \Sigma$.

If the equation (E'') satisfies the conditions :

- 1) There exist a constant $r \geq 0$ and a couple of continuous functions $A : \mathbf{R}_a \rightarrow \mathbf{R}_+$ and $B : \mathcal{T} \rightarrow \mathbf{R}_+$ such that :

- a) $\|k(t, t, x)\| \leq A(t)$ and $\|\mathcal{T}(t, x)\| \leq r, \forall (t, x) \in \mathbf{R}_a \times \Sigma$;
- b) $\|k'_t(t, s, x)\| \leq B(t, s), \forall (t, s, x) \in \mathcal{T} \times \Sigma$;

$$c) \quad \int_a^\infty A(t) dt + \int_a^\infty \int_a^t B(t, s) ds dt = r' < \infty ;$$

- 2) K meets the hypotheses of the preceding proposition ;

- 3) $r' + Cr \leq \rho$,

then it admits at least a solution $x \in S_c$.

PROOF. This corollary follows from the above theorem taking

$$(\mathcal{F}_2 x)(t) = F(t, x(s)) \quad \text{and} \quad (\mathcal{F}_1 x)(t) = \int_a^t k(t, s, x(s)) ds.$$

Indeed, the hypotheses imposed on K and $\mathcal{F}_2$ in the mentioned theorem are satisfied and the ones imposed on the operator $\mathcal{F}_1$ are verified thus :

Let $x \in S$ and

$$y(t) = \int_a^t k(t, s, x(s)) ds, \quad (t \geq a).$$

Taking into account condition 1) we get:

$$\int_a^{\infty} \|y'(t)\| dt \leq r'.$$

This, proves that $(\mathcal{F}_1 S) \subset A_c$.

It still remains to show that $(\mathcal{F}_1 S)$ is equicontinuous on any finite interval $[a, b] \subset \mathbf{R}_a$. Let $\tau \leq t$ and $x \in S$. We have:

$$(1) \quad \begin{aligned} \|(\mathcal{F}_1 x)(t) - (\mathcal{F}_1 x)(\tau)\| &\leq \int_a^b \|k(t, s, x(s)) - k(\tau, s, x(s))\| ds + \\ &+ \int_{\tau}^t \|k(t, s, x(s))\| ds. \end{aligned}$$

From the continuity of k on the compact $\mathcal{T}_b \times \Sigma$ it follows the uniform continuity of this function on the mentioned compact. Hence, for $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that,

$$(2) \quad \sup_{D_\varepsilon} \|k(t, s, x) - k(\tau, s, x)\| < \varepsilon,$$

where $\sup$ refers to the set D_ε of the points (t, s, x) and (τ, s, x) in $\mathcal{T}_b \times \Sigma$ such that $t - \tau < \delta_\varepsilon$.

From condition 1-b) it follows that

$$\int_s^t \frac{\partial}{\partial \tau} \|k(\tau, s, x)\| d\tau \leq \int_s^t B(\tau, s) d\tau$$

for any $(t, s) \in \mathcal{T}$ and any $x \in \Sigma$.

Taking into account condition 1-a) we get the inequality

$$(3) \quad \|k(t, s, x)\| \leq A(s) + \int_s^t B(\tau, s) d\tau$$

for $\forall (t, s, x) \in \mathcal{T} \times \Sigma$.

Denoting by M the upper bound of the function in the right hand side of the inequality (3) on $\mathcal{T}_b$, we have:

$$\int_t^\tau \|k(t, s, x(s))\| ds \leq M(\tau - t)$$

for $\forall (t, \tau) \in \mathcal{T}$ and $\forall x \in S$.

From here and from relations (1) — (3) it results that for $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for every $t, \tau \in [a, b]$ with $t - \tau < \delta_\varepsilon$ we have:

$$\|(\mathcal{F}_1 x)(t) - (\mathcal{F}_1 x)(\tau)\| \leq (b - a) \varepsilon + M \delta_\varepsilon.$$

Taking now $\delta_\varepsilon < \varepsilon$, we get that $\mathcal{F}_1 S$ is a equicontinuous family on every finite interval in $\mathbf{R}_a$.

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R E F E R E N C E S

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DESPRE EXISTENȚA SOLUȚIILOR UNOR ECUAȚII INTEGRALE VOLTERRA ÎN SPAȚIUL FUNCȚIONAL A_c

(Rezumat)

În spațiul funcțional A_c (spațiul Banach al funcțiilor absolut continue și cu derivată sumabilă pe semiaxa $[a, \infty)$) se dau câteva teoreme de existență pentru soluțiile unor clase de ecuații integrale de tip Volterra.

DOUĂ OBSERVAȚII ASUPRA VARIABILEI χ^2

DE

ADRIAN CORDUNEANU

1. Scopul acestei Note este de a da o demonstrație elementară pentru formula densității de probabilitate a variabilei aleatoare χ^2 și de a indica un exemplu foarte natural de astfel de variabilă.

Este bine cunoscut că χ^2 se definește prin relația $\chi^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_n^2$, unde ξ_i ($i = 1, 2, \dots, n$) sînt variabile normale independente, avînd parametrii 0 și σ ; n se numește numărul gradelor de libertate. Deci toate aceste variabile ξ_i au aceeași densitate de probabilitate; dacă notăm

cu ρ_i densitatea lui ξ_i , ($i = 1, 2, \dots, n$) putem scrie $\rho_i(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ pentru $-\infty < x < +\infty$.

Notînd cu ρ densitatea lui χ^2 , vom demonstra (nefăcînd apel la noțiunea de funcție caracteristică) că

$$(1) \quad \rho(x) = \begin{cases} 0 & \text{dacă } x \leq 0 \\ \frac{1}{2^{\frac{n}{2}} \sigma^n \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2\sigma^2}} & \text{dacă } x > 0. \end{cases}$$

În adevăr, dacă $n = 2$, adică $\chi^2 = \xi_1^2 + \xi_2^2$, rezultă

$$(2) \quad \rho(x) = \int_{-\infty}^{+\infty} \tilde{\rho}_1(y) \tilde{\rho}_2(x-y) dy, \quad -\infty < x < +\infty,$$

unde $\tilde{\rho}_1$ și $\tilde{\rho}_2$ sînt densitățile variabilelor ξ_1^2 , respectiv ξ_2^2 . Am folosit faptul că densitatea sumei a două variabile aleatoare independente este egală cu produsul de convoluție a densităților termenilor. Dar deoarece

$$(3) \quad \tilde{\rho}_1(x) = \tilde{\rho}_2(x) = \begin{cases} 0 & \text{dacă } x \leq 0 \\ \frac{1}{\sigma\sqrt{2\pi x}} e^{-\frac{x}{2\sigma^2}} & \text{dacă } x > 0 \end{cases}$$

după cum se poate constata ușor, din formula (2) rezultă că avem $\rho(x) = 0$ pentru $x \leq 0$ și

$$\begin{aligned} \rho(x) &= \int_0^x \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi(x-y)}} e^{-\frac{x-y}{2\sigma^2}} dy = \frac{e^{-\frac{x}{2\sigma^2}}}{\sigma^2 2\pi} \int_0^x \frac{dy}{\sqrt{y(x-y)}} = \\ &= \frac{e^{-\frac{x}{2\sigma^2}}}{\sigma^2 2\pi} \int_0^1 \frac{du}{\sqrt{u(1-u)}} = B\left(\frac{1}{2}, \frac{1}{2}\right) \frac{e^{-\frac{x}{2\sigma^2}}}{\sigma^2 2\pi} = \frac{1}{2\sigma^2 \Gamma(1)} e^{-\frac{x}{2\sigma^2}} \quad \text{pentru } x > 0, \end{aligned}$$

adică în cazul $n = 2$ formula (1) se verifică. Am folosit mai sus cunoscuta relație dintre integralele euleriene

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

și faptul că $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Presupunem acum că pentru un anume număr natural k formula (1) este adevărată. Vom arăta că în acest caz ea se menține și pentru $k + 1$. În adevăr, dacă $\chi^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_k^2 + \xi_{k+1}^2$, putem în mod evident să scriem

$$(4) \quad \rho(x) = \int_{-\infty}^{+\infty} \rho^*(y) \tilde{\rho}_{k+1}(x-y) dy, \quad -\infty < x < +\infty,$$

unde prin ρ^* am notat pentru moment densitatea variabilei $\xi_1^2 + \xi_2^2 + \dots + \xi_k^2$, iar prin $\tilde{\rho}_{k+1}$ am notat densitatea lui ξ_{k+1}^2 . Ținînd seama de ipotezele făcute, din formula (4) rezultă că avem $\rho(x) = 0$ pentru $x \leq 0$ și

$$\begin{aligned} \rho(x) &= \int_0^x \frac{1}{2^{\frac{k}{2}} \sigma^k \Gamma\left(\frac{k}{2}\right)} y^{\frac{k}{2}-1} e^{-\frac{y}{2\sigma^2}} \frac{1}{\sigma \sqrt{2\pi} (x-y)} e^{-\frac{x-y}{2\sigma^2}} dy = \\ &= \frac{e^{-\frac{x}{2\sigma^2}}}{2^{\frac{k+1}{2}} \sigma^{k+1} \Gamma\left(\frac{k}{2}\right) \sqrt{\pi}} \int_0^x (y^{\frac{k}{2}-1} x-y)^{-\frac{1}{2}} dy = \frac{1}{2^{\frac{k+1}{2}} \sigma^{k+1} \Gamma\left(\frac{k+1}{2}\right)} x^{\frac{k+1}{2}-1} e^{-\frac{x}{2\sigma^2}} \end{aligned}$$

pentru $x > 0$, adică tocmai ceea ce trebuia demonstrat.

Conform principiului inducției matematice, conchidem că formula (1) este adevărată, oricare ar fi numărul natural n .

Media și dispersia variabilei χ^2 pot fi de asemenea calculate imediat, plecând direct de la definiția lor.

2. Presupunem că într-un volum Ω se află un anumit gaz, asupra moleculelor căruia nu se exercită nici o forță din afară adică, cu alte cuvinte, ele se află în mișcare haotică. Dacă ne fixăm un reper în spațiu, ai cărui versori îi vom nota ca de obicei cu $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, viteza unei molecule este un vector $\mathbf{v} = \xi_1 \mathbf{i} + \xi_2 \mathbf{j} + \xi_3 \mathbf{k}$, unde proiecțiile pe cele trei axe sînt variabile aleatoare independente. Se poate arăta că ele sînt de tip normal, cu parametrii 0 și σ (σ este același pentru toți ξ_i din motive de simetrie).

Deoarece $v^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$, este evident că pătratul vitezei moleculei este variabila χ^2 cu 3 grade de libertate. Prin urmare, densitatea de probabilitate a lui v^2 este

$$(5) \quad \rho_{v^2}(x) = \begin{cases} 0 & \text{dacă } x \leq 0 \\ \frac{1}{\sigma^3 \sqrt{2\pi}} \sqrt{x} e^{-\frac{x}{2\sigma^2}} & \text{dacă } x > 0. \end{cases}$$

Dacă considerăm v ca o variabilă aleatoare și notăm densitatea sa de probabilitate cu ρ_v , vom găsi că

$$(6) \quad \rho_v(x) = \begin{cases} 0 & \text{dacă } x \leq 0 \\ \frac{2}{\sigma^3 \sqrt{2\pi}} x^2 e^{-\frac{x^2}{2\sigma^2}} & \text{dacă } x > 0. \end{cases}$$

Formula precedentă poartă numele de legea lui Boltzmann-Maxwell și este demonstrată de obicei pe o altă cale.

În sfîrșit, dacă considerăm energia cinetică a unei molecule $E = \frac{mv^2}{2}$ ca o variabilă aleatoare (prin m am notat masa moleculei) și notăm cu ρ_E densitatea sa de probabilitate, printr-un raționament standard se găsește că

$$(7) \quad \rho_E(x) = \begin{cases} 0 & \text{dacă } x \leq 0 \\ \frac{2}{m^2 \sigma^3 \sqrt{\pi}} \sqrt{x} e^{-\frac{x}{m\sigma^2}} & \text{dacă } x > 0. \end{cases}$$

Valoarea medie a energiei cinetice a unei molecule, adică media variabilei E , este

$$(8) \quad M(E) = \int_{-\infty}^{+\infty} x \rho_E(x) dx = \int_0^{+\infty} x \rho_E(x) dx = \frac{3m\sigma^2}{2}.$$

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TWO REMARKS ABOUT THE RANDOM VARIABLE χ^2

(Summary)

The aim of this Note is to give an elementary proof (no making use of the notion of characteristic function) for the formula (1) and also to indicate a very natural example of random variable χ^2 .

CONSIDERATIONS ON MOMENTS AND FACTORIAL MOMENTS OF A RANDOM VARIABLE

BY

GH. FLOREA

In mathematical statistics there are many problems in which it is difficult or even impossible to determine the distribution function of the random variables. Such being the situation, often, and not without utility, one succeeds to describe on the whole the distribution of the random variable by means of moments or by means of central moments or factorial moment as well as by means of some functions of these moments.

Let X be a random variable whose distribution function we denote with $F(x)$. We define the moment of n order of X random variable, denoted $\mu_n(x)$, as being

$$(1) \quad \mu_n(x) = \int_{-\infty}^{\infty} x^n dF(x).$$

In problems connected with discrete random variables often, it is better to find the moments after the determination of the so called factorial moments, taking into account that the last ones are easier to determine.

If we denote

$$(2) \quad x^{[n]} = x(x-1)(x-2) \dots (x-n+1),$$

then the factorial moment of the n order of X random variable, $\mu_{[n]}(x)$, is defined by

$$(3) \quad \mu_{[n]}(x) = \int_{-\infty}^{\infty} x^{[n]} dF(x).$$

Let recall that in the case of some discrete random variables, one determines easier the factorial moments; then in order to get the moments of random variables after a previous determination of the factorial moments, we have to determine a recurrence formule expressing the function moments by means of factorial moments. For this purpose we shall express x^n as a function of $x^{[1]}, x^{[2]}, \dots, x^{[n]}$ by using Newton's interpolation formula for polynomials:

$$(4) \quad x^n = (x^n)_0 + \frac{(\Delta x^n)_0}{1!} x^{[1]} + \frac{(\Delta^2 x^n)_0}{2!} x^{[2]} + \dots + \frac{(\Delta^n x^n)_0}{n!} x^{[n]},$$

where Δ^k means the finite difference of k order of the polinomial x^n . With this formula we get:

$$(5) \quad \begin{aligned} x^n = & x^{[1]} + \frac{x^{[2]}}{2!} \sum_{i_1=1}^{n-1} C_n^{i_1} + \frac{x^{[3]}}{3!} \sum_{i_1=1}^{n-2} C_n^{i_1} \sum_{i_2=1}^{(n-1)-i_1} C_{n-i_1}^{i_2} + \dots + \\ & + \frac{x^{[n]}}{n!} \sum_{i_1=1}^1 C_n^{i_1} \sum_{i_2=1}^{2-i_1} C_{n-i_1}^{i_2} \sum_{i_3=1}^{3-i_1-i_2} C_{n-i_1-i_2}^{i_3} \dots \sum_{i_{n-1}=1}^{(n-1)-i_1-\dots-i_{n-2}} C_{n-i_1-\dots-i_{n-2}}^{i_{n-1}}. \end{aligned}$$

By substituting (5) in (1) we obtain the required recurrence formula

$$(6) \quad \begin{aligned} \mu_n = & \mu_{[1]} + \frac{\mu_{[2]}}{2!} \sum_{i_1=1}^{n-1} C_n^{i_1} + \frac{\mu_{[3]}}{3!} \sum_{i_1=1}^{n-2} C_n^{i_1} \sum_{i_2=1}^{(n-1)-i_1} C_{n-i_1}^{i_2} + \\ & + \frac{\mu_{[4]}}{4!} \sum_{i_1=1}^{n-3} C_n^{i_1} \sum_{i_2=1}^{(n-2)-i_1} C_{n-i_1}^{i_2} \sum_{i_3=1}^{(n-1)-i_1-i_2} C_{n-i_1-i_2}^{i_3} + \dots + \\ & + \frac{\mu_{[n]}}{n!} \sum_{i_1=1}^1 C_n^{i_1} \sum_{i_2=1}^{2-i_1} C_{n-i_1}^{i_2} \sum_{i_3=1}^{3-i_1-i_2} C_{n-i_1-i_2}^{i_3} \dots \sum_{i_{n-1}=1}^{(n-1)-i_1-\dots-i_{n-2}} C_{n-i_1-\dots-i_{n-2}}^{i_{n-1}}. \end{aligned}$$

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CONSIDERAȚII ASUPRA MOMENTELOR ȘI MOMENTELOR FACTORIALE ALE UNEI VARIABILE ALEATOARE

(Rezumat)

Se stabilește o formulă de recurență ce exprimă legătura dintre momentele și momentele factoriale ale unei variabile aleatoare X .

CONSTRUCȚIA UMBRELOR ÎN DIEDRUL DE REFERINȚĂ DE UNGHII VARIABIL ¹⁾

DE

I. RUSU

1. Introducere

În lucrări anterioare autorul s-a ocupat cu studiul reprezentării plane a elementelor și corpurilor geometrice, precum și a unor obiecte, considerînd un sistem de referință format dintr-un diedru ale cărui semiplane fac un unghi arbitrar. Unul din plane este orizontal de cotă zero numit planul reprezentării H , iar al doilea este planul de proiecție P , ce se intersectează cu primul după dreapta $\Delta \equiv Ox$.

În această lucrare se prezintă construcția umbrelor reale și virtuale pe planele diedrului de referință ale elementelor geometrice și ale unor figuri plane frecvent întîlnite în toate domeniile. Realizarea construcției plane a umbrelor se face aici pe baza corespondenței de omologie afină ce se stabilește între umbre și proiecțiile elementelor sau obiectelor de pe planul reprezentării.

2. Umbra punctului

Fie punctul $A(a; Aa)$, în care $Aa = Z_A$. După cum se observă din fig. 1, poziția punctului A considerat poate fi determinată în sistemul de referință considerat prin coordonatele sale cilindrice astfel: distanța OQ_{18} — abscisa punctului, $AQ_{18} = \rho$ — distanța punctului la axa Δ și unghiul θ pe care raza ρ îl face cu planul H al reprezentării. Deci dacă: $OQ_{18} = x$, $AQ_{18} = \rho$ și $AQ_{18}a = \theta$, atunci punctul A este determinat prin coordonatele $A(x, \rho, \theta)$. Se proiectează ortogonal punctul A pe pla-

1) Prezentată la sesiunea științifică a Institutului politehnic din Iași, din 2—4 iulie 1970.

nele diedrului de referință. Astfel, în planul H se obține proiecția a a punctului, iar pe planul P proiecția A_1 . Aceste proiecții se obțin prin intersecția urmelor planului proiectant al punctului A cu proiectantele aceleiași punct. Se consideră apoi proiectanta D a punctului A , de o direcție arbitrară, ca rază de lumină. Intersecția ei cu planul de proiecție P reprezintă umbra Ap , care este un punct.

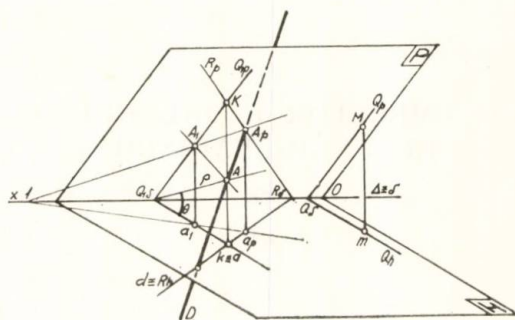


Fig. 1

Punctul umbră Ap s-a obținut astfel: s-a considerat planul proiectant R al razei de lumină D astfel că $R \equiv [(D) \cap (Aa)]$ (planul R este vertical). Prin urmare urma Rh Ua , în care $U = (D) \cap [H]$. Urma Rp este determinată de R și de K , în care $R_8 = (Rh) \cap (\Delta)$ și $K = (Aa) \cap [P]$.

Deci, punctul umbră Ap este intersecția razei de lumină $D(d, D_0)$ cu urma Rp a planului proiectant R din planul P . Operațiile de proiectare și de intersectare se fac în planele proiectante ale elementelor geometrice în cauză, deci pe urmele acestora. În epura din fig. 2 a fost necesar ca aceste plane să fie rabătute pe planul reprezentării. Astfel, planul Q_1 al punctului $A(a; 0,5)$ rabătut a condus la găsirea proiecțiilor ortogonale A_1 și a_1 ale acestui punct pe planele P și respectiv H . De aceea prin Q_{18}^m s-a dus urma $O_{1p} \parallel O_p$, iar apoi din A rabătut s-a dus $AA_1 \perp Q_{1p}$, iar $AK \perp Q_{1h}$. Umbra Ap s-a găsit în rabatere ca punct de intersecție dintre D_0 și R_{1p} . Proiecția ortogonală ap a acesteia pe planul reprezentării H se obține dacă se duce o perpendiculară prin Ap la urma Rh . Prin umbra Ap ($ap; z_{Ap}$) se consideră planul proiectant Q_2 , ce se rabate în planul reprezentării. În acest scop se duce urma $Q_{2p} O_h$, iar segmentul $a_p A_p = z_{Ap}$. Prin rabaterea planului P în jurul dreptei Δ , într-un sens sau altul, se găsesc punctele A_{10} și A_{20} precum și A_{1p} , A_{2p} . Primele puncte sînt pozițiile după rabaterea proiecției ortogonale a punctului A pe planul P , iar celelalte reprezintă pozițiile punctului umbră după efectuarea aceleiași rabateri. Așa cum se observă, între proiecțiile a_1 și a_p și punctele A_1 și A_p în

raza de lumină R paralelă cu direcția $\bar{R}$. Prin urmare planul de lumină L este determinat de $[L] = (D) \cap (R)$.

Intersecțiile planului de lumină cu planele diedrului de referință conduc la găsirea urmelor acestuia, care sînt de fapt umbrele dreptei D pe aceste plane, astfel: $(D_h) = [L] \cap [H]$, $(D_p) = [L] \cap [P]$.

În fig. 3 b, dreapta D trece printr-un punct dat $A(a; z_A)$ și este paralelă cu direcția $\bar{D}$, sau dacă se exclude direcția $\bar{D}$ atunci dreapta considerată poate să fie determinată fie de două puncte neconfundate, așa cum s-a menționat, fie de un punct al ei, de proiecția d pe planul H al reprezentării, precum și de unghiul pe care dreapta îl face cu planul reprezentării. Prin urmare: $D \equiv \{A(a; z_A); d, \varphi\}$. Și în cazul a și în cazul b , ale fig. 3, urmele planului de lumină, umbrele dreptei, trec prin urmele U_h și U_p ale acesteia, pentru că este știut că umbra unui punct pe un plan, în care acesta este situat, este punctul însuși. Pentru a trasa umbrele D_h și D_p se utilizează umbra U_r care rezultă aici din intersecția razei de lumină R cu planul H , deci: $U_r = (R) \cap [H]$, astfel că $D_h \equiv (U_h, U_r)$ și $D_p \equiv (U_p, U_\delta)$, unde U_δ este punctul în care planul de lumină in-

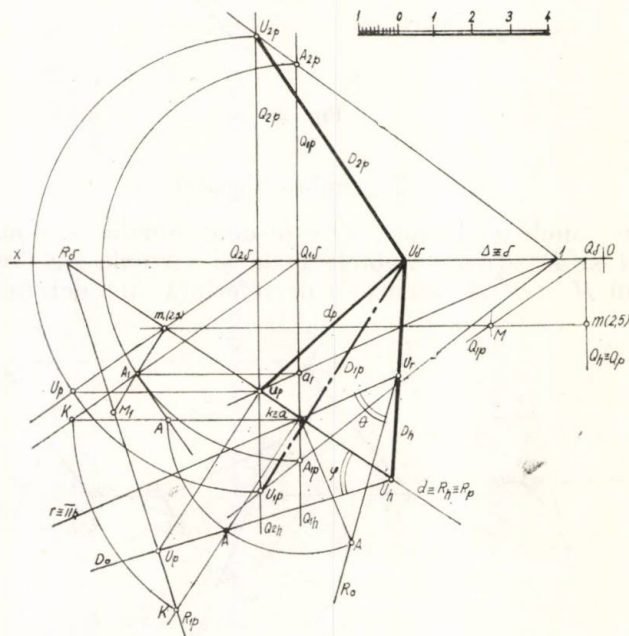


Fig. 4

tersectează axa de referință $\Delta \equiv (Ox)$ sau, practic, punctul în care urma D_h intersectează această axă. Prin dreapta D s-a considerat planul ajutător $R = (D) \cap (Aa)$, care este plan vertical, pentru că $(Aa) \perp [H]$. Pentru găsirea urmelor U_h și U_p atît în schița din fig. 3 b cît și în epura din fig. 4, prin rabatarea planului R pe planul reprezentării se găsește la intersecția dreptei D cu proiecția sa d urma $U_h = u_h$. Prin urmare se scrie $U_h = (D) \cap d$. Pentru găsirea urmei U_p se consideră că aceasta este situată pe

urma R_p a planului ajutător. Trasarea acestei urme, atît în spațiu cît și în epură, necesită cunoașterea a două puncte ce-i aparțin. Unul din ele este $R_8 = (\Delta) \cap (R)$, sau, așa cum se observă, punctul R se găsește la intersecția urmei R_h cu axa Δ . Al doilea punct este K , situat pe proiectanta punctului A , proiectantă rezultată din intersecția planelor verticale R și Q_1 . Planul Q_1 trece prin A și este perpendicular pe (Δ) . Evident că o dată cu planul R se rabată în planul H și elementele conținute, deci și proiectanta Aa împreună cu punctul K (fig. 4) ajungînd în poziția R_{1p} . Razele de rabatare pentru punctele A și K sînt cotele acestora $Z_A = Aa$ și $Z_K = Kk$. Deci, $(R_{1p}) \equiv (K; R_8)$. La intersecția acestei urme R_{1p} cu dreapta D rabătută în planul H al reprezentării se găsește urma U_p , care ridicată împreună cu planul R în poziția inițială conduce la găsirea proiecției ei pe planul H . În epură acest lucru se realizează ducînd prin U_p o perpendiculară pe $d \equiv (R_h)$. Piciorul perpendicularei pe d este proiecția u_p menționată. Rabătînd planul P pe planul H împreună cu U_p , ca punct conținut, se găsește soluțiile U_{1p} și U_{2p} , după cum rabaterea se face într-un sens sau celălalt în jurul axei Δ . Rabaterea urmei U_p s-a făcut pe baza triunghiului de poziție $U_p u_p O_{2s}$ în care: $U_p u_p = Z_{u_p}$ — cota acestui punct și una din catete, $U_p O_{2s}$ — ipotenuza acestui triunghi, care reprezintă distanța punctului U_p la axa Δ și care se mai poate nota cu r sau cu ρ și cea de-a doua catetă $u_p O_{2s} = \rho \cos \varphi$.

Construcția în epură a urmei U_r , ca punct de intersecție dintre raza de lumină R și planul reprezentării H , este analogă cu cea indicată pentru urma U_p . S-a considerat că raza R și proiectanta Aa determină un plan vertical Π astfel că atît în spațiu cît și în epură $(\Pi_h) \equiv \gamma$. Cunoșcînd unghiul θ , pe care raza R îl face cu planul H , se rabate planul Π pe planul H împreună cu elementele conținute. În epură se duce prin proiecția a o perpendiculară la $r \equiv (\Pi_h)$, pe care se măsoară din a distanța $Z_A = Aa$. Apoi prin A se duce o dreaptă care face cu proiectanta Aa unghiul $90^\circ - \theta$, care este raza vizuală R_0 rabătută pe planul H . La intersecția acestuia cu proiecția r de pe același plan se găsește urma U_r . Deci umbra dreptei pe planul H este $(D_h) \equiv (U_h; U_r)$, care intersectează dreapta în punctul U . Unind prin drepte acest punct cu punctele U_{1p} și U_{2p} se găsește umbrele D_{1p} și D_{2p} ale dreptei pe planul P , rabătute în planul H . Același punct N unit printr-o dreaptă cu proiecția u_p determină proiecția acestei umbre pe planul H al reprezentării.

4. Umbra figurilor plane

4.1. Umbra unui poligon situat într-un plan de nivel

Se consideră dreptunghiul $ABCE$ situat într-un plan de nivel H_1 de cotă oarecare. În epură acest poligon este dat prin proiecția sa $abce$ de pe planul H al reprezentării (fig. 5). Fie raza de lumină $R(r, R_0)$, ce trece prin vîrfurile $A(a)$ al poligonului plan dat. Considerînd că între acest poligon și umbrele sale din planele H și P ale diedrului de referință se pot stabili corespondențe de afinitate, se poate trece la construcția umbrelor, care vor fi figuri afine ale dreptunghiului dat.

În mod obișnuit planul P este determinat de $(Ox) \equiv (\Delta)$ și de punctul $M(m, Z_M)$. Spre a fi determinată afinitatea este necesar să se găsească cu ajutorul razei R umbrele vîrfurilor A . Pentru exemplificare s-a considerat că planul de nivel al dreptunghiului dat are cota $Z_{H1} = 2,5$ unități. Pentru găsirea umbrelor punctului A raza de lumină se intersectează cu planele H și P . În acest scop se consideră planul Π dus prin raza de lumină R , plan perpendicular pe planul reprezentării H . Acest plan este reprezentat prin urmele $\Pi_h \equiv \Pi_p \equiv r$. Urma Π_p este rabătută în planul H al reprezentării obținîndu-se dreapta $\Pi_{1p} \equiv (\Pi_s, N)$. Se observă că $Z_N = Z_M$, deoarece aceste puncte aparțin aceleiași orizontale a planului P . Evident că axa de afinitate dintre planul H_1 al dreptunghiului dat și umbra sa pe planul P este dreapta $O_1 X_1 \equiv \Delta_1$, ca intersecție a celor două plane astfel: $(Ox) = [H] \cap [P]$, $(O_1 x_1) = [H_1] \cap [P]$; dacă $[H] \parallel [H_1] \Rightarrow$

$\Rightarrow (Ox) \parallel (O_1x_1)$. Este deci suficient să se cunoască un punct al axei O_1x_1 , pentru ca ea să fie determinată și deci trasată. Or, rabaterea planului R în planul H împreună cu elementele conținute permite găsirea acestui punct:

$$\begin{cases} (H_1R) = [H_1] \cap [R] \\ (R_{1p}) = [P] \cap [R] \end{cases} \Rightarrow (H_1R) \cap (R_{1p}) = \Omega,$$

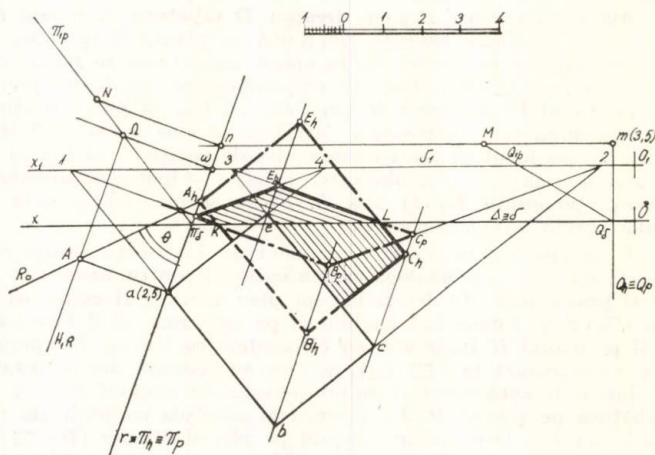


Fig. 5

care este un punct comun planelor H_1 și P , deci punctul căutat. În epură punctul Ω din rabatere este proiectat ortogonal pe planul H în $\omega \equiv r \equiv (Rh) \equiv (Rp)$. Prin proiecția ω se duce axa $O_1x_1 \equiv \Delta_1 \equiv \delta_1$, paralelă la axa Ox . Raza de lumină R este rabătită în același plan al reprezentării H , cu ajutorul virfului A și al unghiului O . Astfel, poziția rabătită R_0 a razei de lumină intersectează proiecția r în A_h , care este umbra acestui punct pe planul H și urma R_{1p} în A_p , punct ce este umbra aceluiași virf pe planul P . Umbra dreptunghiului $ABCE$ pe planul H se obține considerind razele de lumină pentru fiecare virf (ca paralele la r); din A_h se duc paralele la laturile proiecției dreptunghiului, ce sînt intersectate fiecare cu proiectanta corespunzătoare. În acest mod se obține dreptunghiul $A_hB_hC_hE_h$ congruent cu cel dat $ABCE$ ($abce$). Unghiul pe planul P a aceluiași dreptunghi se construiește pe baza corespondenței de afinitate ce se stabilește între acest dreptunghi și umbra sa. Se obține paralelogramul $A_pB_pC_pE_p$ considerind ca și mai înainte ca axă de afinitate, dreapta $O_1x_1 \equiv \Delta_1 \equiv \delta_1$. Umbrele pe semipanele H_a și P_s s-au considerat reale deoarece ele sînt văzute. Pe semipanele H_p și P_t aceste umbre sînt virtuale (fiind nevăzute). Linia de fringere a umbrelor reale și virtuale este dreapta $KL \in Ox$.

4.2. Umbra cercului situat într-un plan vertical

Fie cercul $\Omega(\omega)$ situat în planul vertical $R(R_h, R_p)$ (fig. 6). Planul de proiecție P este determinat, ca și în alte cazuri, de axa Ox și de un punct $M(m, Z_m)$. Se consideră că între cercul dat și umbrele acestuia pe planele diedrului de referință există corespondențe de afinitate, care pot fi utilizate în scopul construcției umbrelor cercului Ω după cum urmează. Cu ajutorul razei de lumină $R(r, R_0)$, ce trece prin centrul Ω , se găsesc direct în epură umbrele acestui punct. În acest sens se rabate planul R pe

planul reprezentării H împreună cu cercul dat. În rabatere se consideră diametrii principali AB și CE . Atît centrul Ω cît și extremitățile diametrilor se proiectează ortogonal pe urma $Rh \equiv Rp$, operația corespunzînd ridicării planului R . Se duce prin $R(r, R_0)$ planul Π vertical astfel încît $r \equiv (\Pi_h) \equiv (\Pi_p)$. Rabătînd planul Π tot pe planul reprezentării H împreună cu elementele conținute se obține raza de lumină R_0 și urma Π_{1p} . Umbrele lui sînt: $\Omega_h = (R_0) \cap (\Pi_h)$ pe planul reprezentării H , $\Omega_p = (R_0) \cap (\Pi_p)$ pe planul de proiecție P .

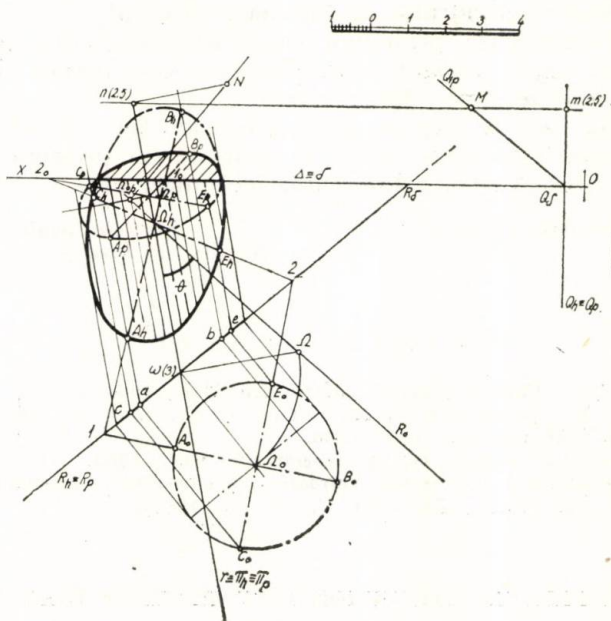


Fig. 6

Evident că umbra se proiectează ortogonal pe urma $\Pi_h \equiv \Pi_p$, ceea ce corespunde ridicării planului Π , obținîndu-se proiecția Ω_{1p} în planul H . Construcția umbrei cercului Ω pe planul H revine la determinarea umbrelor pe acest plan a diametrilor principali considerați. Axa de afinitate fiind urma Π_h , se prelungește acești diametri din rabatere, ce intersectează axa menționată în punctele 1 și respectiv în 2. Aceste puncte se unesc prin linii drepte cu umbra Ω_h obținînd astfel liniile afine ale diametrilor. Prin punctele a, b, c și e se duc proiecțiile razelor de lumină ca drepte paralele la r , care se intersectează cu liniile afine corespunzătoare în punctele A_h, B_h, C_h și E_h . Diametrii conjugăți $A_h B_h$ și $C_h E_h$ conduc la construcția elipsei Ω_h , curbă afină a cercului Ω și care este umbra Ω pe planul H a acestuia. Diametrii conjugăți afini intersectează axa Ox în punctele I_0 și 2_0 . Între umbrele cercului Ω de pe planul H și din planul P se stabilește de asemenea o corespondență de afinitate în care axa Ox este axă de afinitate. Se unesc prin linii drepte punctele I_0 și 2_0 , cu umbra proiecției Ω_{1p} . Aceste drepte afine sînt intersectate de aceleași proiecții ale razelor de lumină utilizate și pentru obținerea umbrei Ω_h , în punctele A_p, B_p, C_p , și E_p . Diametrii conjugăți A_p, B_p și C_p, E_p permit trasarea elipsei Ω_{1p} , umbra cercului dat pe planul P . Ca și în cazul figurii precedente, s-a considerat că pe semiplanele H_a și P_s elipsele umbră sînt reale, iar pe semiplanele H_p și P_t sînt virtuale.

5. Concluzii

1. Se introduce un diedru de referință cu unghi arbitrar, diedrul ortogonal reprezentînd un caz particular al acestuia.

2. Construcția umbrelor s-a făcut pe baza corespondenței de afinitate între elementul sau figura geometrică dată și umbrele respective. În acest scop s-a considerat sursa de lumină ca punct impropriu, ceea ce a condus la utilizarea cilindrului de lumină proiectant.

3. Se generalizează principiul construcției umbrelor într-un diedru de referință de unghi variabil, corespondența considerată simplificînd în mod sensibil reprezentările plane necesare.

4. Metoda care se expune pentru construcția umbrelor elementelor și figurilor geometrice plane poate fi utilizată în proiectare în domenii foarte variate (construcții, arhitectură, mecanică, etc).

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SCHATTEN-KONSTRUKTION IN VARIABLEM DIEDER

(Zusammenfassung)

In dem Aufsatz, ist die Konstruktion der reellen und virtuellen Schatten auf die Ebenen des in betracht genommenen Referenz-Dieders realisiert worden. In diesem Sinne wurde die affine Homologie-Korrespondenz zwischen den zu konstruierenden Schatten, und den Elementen oder den im Raum gegebenen und in der Epure, durch ihre orthogonalen Projektionen in Ebene II, dargestellten geometrischen Figuren gebraucht.

Der Übergang zur Epure (zu einer einzigen Ebene), wurde gewöhnlich durch rotieren (rabatieren) der Ebene, P , um der Axe $\Delta \equiv (Ox)$, zusammen mit den enthaltenen Elementen zustandegebracht, so dass $[P_1] \equiv [II]$; es wurden in dieser Weise je zwei Verhältnisse, sowohl für die Projektionen, als auch für die Schatten gefunden.

SUR LE PROBLÈME DE LA LOCALISATION EN MÉCANIQUE QUANTIQUE

PAR

ROBERT VALLÉE

1. Opérateur carré de la distance dans le cas classique

En mécanique quantique les opérateurs coordonnées x_1, x_2, x_3 , d'un corpuscule tel que l'électron, ont chacun un spectre continu de valeurs propres qui est $\mathcal{R}$ tout entier. Ces opérateurs peuvent être remplacés par un opérateur de position unique, de nature vectorielle, $\vec{x} = \vec{OM}$. Le problème de localisation est évidemment lié à celui de l'évaluation de la distance du corpuscule à un point donné O . Il est donc naturel de s'intéresser à l'opérateur carré de la distance à un point. Cet opérateur $\vec{x}^2 = \vec{OM}^2 = r^2$ agit multiplicativement, il est hermitien, ses distributions propres (il n'admet pas de fonctions propres) ont chacune pour support une sphère de centre O ou O lui-même. Son spectre est continu, c'est l'ensemble $\mathcal{R}_+$ des réels positifs (zéro compris). Le spectre de l'opérateur distance, ensemble des racines carrées positives des éléments de $\mathcal{R}_+$, est donc classiquement $\mathcal{R}_+$.

Soit ψ une fonction de $\vec{x}$ (ou éventuellement une distribution) possédant une transformée de Fourier ($\mathcal{F}\psi = \varphi$, fonction de $\vec{p}$, définie sur l'espace vectoriel de quantités de mouvement. Dans le cas où ψ est une fonction on peut écrire

$$(1.1) \quad \varphi(\vec{p}) = h^{-\frac{3}{2}} \iiint_{\mathcal{R}^3} \psi(\vec{x}) e^{-i \frac{2\pi}{h} \vec{p} \cdot \vec{x}} dx_1 dx_2 dx_3.$$

S'il s'agit de la fonction d'onde $\Psi(\vec{x}, t)$ on a de façon analogue

$$(1.2) \quad \Phi(\vec{p}, t) = h^{-\frac{3}{2}} \iiint_{\mathbb{R}^3} \Psi(\vec{x}, t) e^{-i \frac{2\pi}{h} \vec{p} \cdot \vec{x}} dx_1 dx_2 dx_3.$$

La recherche des éléments propres de l'opérateur r^2 peut évidemment être remplacée par celle des éléments propres de l'opérateur $-\frac{h^2}{4\pi^2} \Delta$ où Δ désigne le laplacien) image de Fourier de l'opérateur r^2 (l'image de Fourier de l'opérateur $\vec{x}$ étant $\frac{i\hbar}{2\pi} \vec{\text{grad}}$). On a

$$(\mathcal{F}(r^2\psi)) = -\frac{h^2}{4\pi^2} \Delta\varphi,$$

l'équation $r^2\psi = \lambda\psi$ est donc équivalente à $-\frac{h^2}{4\pi^2} \Delta\varphi = \lambda\varphi$, ψ et par suite φ étant différents de la fonction nulle. Le spectre de $-\frac{h^2}{4\pi^2} \Delta$ est naturellement $\mathcal{R}_+$ et ses fonctions propres sont les transformées de Fourier des distributions propres de r^2 .

2. Opérateur carré de la distance dans le cas d'hypothèses généralisées

Pour éviter certaines difficultés, concernant les opérateurs coordonnées, on a parfois proposé d'imposer à $\Psi(\vec{x}, t)$ d'être telle que la transformée $\Phi(\vec{p}, t)$, donnée par (1.2), soit nulle hors d'un ensemble borné ou satisfasse des hypothèses voisines de celles-ci [1], [2]. On s'est, semble-t-il, plus rarement intéressé aux conséquences que ce genre de conditions entraîne, si on les prend dans toute leur généralité, pour l'opérateur carré de la distance.

Nous considérons un corpuscule libre dont la fonction d'onde Ψ satisfait l'équation

$$(2.1) \quad \frac{\partial \Psi}{\partial t} - i \frac{h}{4\pi m} \Delta \Psi = 0,$$

nous avons donc

$$(2.2) \quad \Phi(\vec{p}, t) = \Phi(\vec{p}, 0) e^{-i \frac{\pi p^2}{hm} t}.$$

Nous faisons l'hypothèse que $\Phi(\vec{p}, 0)$ est nul à l'intérieur et sur le bord d'un compact connexe, avec bord, Ω (il est utile de supposer, de plus, Ω simplement connexe et symétrique par rapport à l'origine I de l'espace

des quantités de mouvement, I appartient alors à Ω). Ces hypothèses, faites sur $\Phi(\vec{p}, 0)$, sont aussi vérifiées par $\Phi(\vec{p}, t)$ d'après la forme de l'équation d'évolution (2.2). Nous considérons donc ces hypothèses comme valables pour toutes les fonctions $\Phi(\vec{p}, t)$ et adoptons les hypothèses équivalentes concernant les fonctions d'onde $\Psi(\vec{x}, t)$ satisfaisant l'équation (2.1).

Nous allons maintenant nous intéresser aux valeurs propres de l'opérateur $-\frac{\hbar^2}{4\pi^2}\Delta$ agissant sur les fonctions $\varphi(\vec{p})$ satisfaisant les conditions imposées, ci-dessus, aux fonctions $\vec{p} \rightarrow \Phi(\vec{p}, t)$. On sait que dans ces conditions l'opérateur $-\frac{\hbar^2}{4\pi^2}\Delta$ est *hermitien*, que ses valeurs propres sont *strictement positives*, qu'elles forment un ensemble *infini dénombrable* (théorème de Dirichlet), *sans points d'accumulation* (conséquence d'un théorème de Fredholm), donc *non borné*. Les fonctions propres permettent de plus la construction d'une base hilbertienne pour les fonctions de carré intégrable sur Ω donc pour les fonctions $\vec{p} \rightarrow \varphi(\vec{p})$ et $\vec{p} \rightarrow \Phi(\vec{p}, t)$. Les valeurs observables du carré de la distance au point O sont les valeurs propres de l'opérateur $-\frac{\hbar^2}{4\pi^2}\Delta$. Elles possèdent donc les propriétés énoncées ci-dessus. Il en est de même de l'ensemble de leurs racines carrées (positives) qui est l'ensemble des valeurs observables de la distance du corpuscule au point O , donc à un point quelconque donné. Il existe par conséquent une valeur observable λ_0 de la distance qui est *différente de zéro et plus petite que les autres*, d'autre part les valeurs observables de la distance, classées dans l'ordre naturel, forment une suite non bornée (sans que ces valeurs soient multiples de λ_0).

3. Anisotropie et isotropie

Deux cas particuliers principaux peuvent être envisagés. Il y a tout d'abord le cas où Ω est un cube dont le centre de symétrie est l'origine I de l'espace des quantités de mouvement. Ce choix introduit une anisotropie dans l'espace des quantités de mouvement et par suite aussi dans l'espace proprement dit. Cette particularité peut être considérée soit comme un inconvénient grave, soit comme un trait spécifique de l'espace de la microphysique, trait qui ne serait plus sensible au niveau macroscopique [7], [8], [9]. Si a est la demi-longueur du côté du cube Ω on sait que les valeurs propres de $-\frac{\hbar^2}{4\pi^2}\Delta$ sont $\frac{\hbar^2}{16a^2}(n_1^2 + n_2^2 + n_3^2)$ et par suite que les valeurs observables de la distance sont

$$(3.1) \quad \frac{h}{4a} \sqrt{n_1^2 + n_2^2 + n_3^2},$$

les n_i étant des entiers positifs dont aucun n'est nul. La valeur minimale λ_0 est alors $\frac{h\sqrt{3}}{4a}$. Les fonctions propres de l'opérateur r^2 s'expriment à l'aide de la fonction $u \rightarrow \frac{\sin u}{u}$ ce qui permet, compte tenu des propriétés de cette fonction, une évaluation facile de la probabilité de voir la distance au point O prendre la valeur (3.1).

L'autre cas particulier, le seul qui n'introduise pas d'anisotropie, est celui où Ω est une boule (euclidienne) fermée, de centre I . Si R est le rayon de cette boule, des calculs classiques montrent que les valeurs propres de l'opérateur $-\frac{h^2}{4\pi^2} \Delta$ sont $\frac{h^2}{4\pi^2 R^2} \sigma_{\sqrt{n^2+1/4}, p}^2$, où n est un entier positif ou nul et $\sigma_{\sqrt{n^2+1/4}, p}$ la p -ième racine strictement positive de la fonction de Bessel $J_{\sqrt{n^2+1/4}}$. Les valeurs observables de la distance sont alors

$$(3.2) \quad \frac{h}{2\pi R} \sigma_{\sqrt{n^2+1/4}, p}$$

et la valeur minimale λ_0 est $\frac{h}{2\pi R} \sigma_{1/2, 1} = \frac{h}{2R}$. Les fonctions propres de $-\frac{h^2}{4\pi^2} \Delta$ s'expriment principalement à l'aide des fonctions de Bessel et d'harmoniques sphériques.

On vérifie sur ces deux exemples que lorsque Ω tend à occuper tout l'espace des quantités de mouvement (a dans le premier cas et R dans le second tendent vers l'infini) la distance minimale λ_0 tend vers zéro et l'ensemble des distances observables tend à s'identifier à $\mathcal{Q}_+$.

4. Remarques finales

L'intervention d'une distance minimale λ_0 , conséquence des hypothèses faites sur les fonctions Ψ , a été invoquée en microphysique en utilisant, comme nous l'avons rappelé plus haut [1], [2], des hypothèses voisines des précédentes appliquées à l'opérateur distance (plus rarement, semble-t-il, à l'opérateur carré de la distance). Elle a été aussi envisagée dans le cadre de la relativité restreinte [3], [4], dans celui de la relativité générale [5], [6], en introduisant un espace (ou un espace-temps) anisotrope [7], [8], [9], enfin d'un point de vue topologique [10].

Le modèle que nous avons considéré est non relativiste. Il n'exclut pas la possibilité d'un espace anisotrope (il faut et suffit pour cela que le bord de Ω ne soit pas une sphère de centre I). Il présente des particularités non absentes, d'ailleurs, des modèles cités ci-dessus. Elles ne conduisent pas à des contradictions si l'on observe que l'existence d'un spectre des distances observables du genre rencontré ici conduit à un type de localisation des particules différent de celui associé au spectre $\mathcal{Q}_+$.

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ASUPRA PROBLEMEI LOCALIZĂRII ÎN MECANICA CUANTICĂ

(Rezumat)

Se studiază spectrul operatorului distanță la un punct în ipoteze generalizate. Valorile proprii sînt strict pozitive, mulțimea lor fiind infinită, numărabilă, nemărginită și fără punct de acumulare; există deci o valoare proprie mai mică decît celelalte și nenulă. Două cazuri particulare ilustrează rezultatele, primul corespunzînd unui spațiu anizotrop, iar al doilea la un spațiu izotrop.

PRINCIPES VARIATIONNELS „ROUTHIENS” ET ÉQUATIONS ROUTHINIENNES DES SYSTÈMES VARIABLES DE PARTICULES

PAR

SOULEYMANE NIANG

1. Introduction

Un principe variationnel correspondant aux formes lagrangiennes des équations de Mangeron-Tzénov a déjà été dégagé pour les systèmes constants [1]. Il s'agit ici, dans le cadre de systèmes variables de particules [2], c'est-à-dire de systèmes matériels „échangeant des particules avec l'extérieur“ d'un principe variationnel appelé routhien, parce qu'il conduit à l'obtention des équations analytiques des mouvements des systèmes variables libres ou liés, qui sont la généralisation des équations de Routh relatives aux systèmes mécaniques constants classiques. Ces équations s'étendent bien entendu aux fusées et aux systèmes dits de masse variable étudiés notamment par R. Raffin [3], J. L. Mériam [4] et plus récemment par N. Irimiciuc [5].

Soit, par rapport à un référentiel inertiel $(\mathcal{R})$, $\mathcal{L}(t) \equiv \{(m_i, \vec{M}_i, \vec{V}_i)_t \mid i \in \mathcal{L}(t)\}$ un système variable de particules (d'indexant $\mathcal{L}(t)$) dont les positions $\vec{M}_i$ dépendent, à l'instant t , de n paramètres $q^\nu(t)$; $\vec{M}_i(t)$ étant exprimable explicitement — et avant toute considération dynamique en fonction des q^ν et de t .

2. Principes variationnels

1. *Les q^ν sont indépendants.* Distinguons alors, parmi les q^ν , l paramètres $q^\lambda (\lambda = 1, 2, \dots, l)$, les $n - l$ autres étant désignés par $q^\mu (\mu = l + 1, \dots, n)$ et posons :

$$(1) \quad p_{\lambda}^* = \frac{\partial T}{\partial \dot{q}^{\lambda}},$$

où T est l'énergie cinétique de $\mathcal{L}(t)$, les p_{λ}^* étant les moments conjugués de Lagrange associés aux q^{λ} .

Les p_{λ}^* sont linéaires par rapport aux $\dot{q}^{\lambda}$, de sorte qu'en général il existe un domaine de dimension l où les $\dot{q}^{\lambda}$ s'expriment linéairement en fonction des p_{λ}^* , soit

$$(2) \quad \begin{cases} \dot{q}^{\lambda} = f(p_{\lambda}^*, \dot{q}^{\mu}, q^{\lambda}, q^{\mu}; t), \\ (\lambda=1, 2, \dots, l; \mu=l+1, \dots, n), \end{cases}$$

f étant linéaire en p_{λ}^* .

Dès lors associons à l'espace E_P du point figuratif $P(q^{\nu})$ de $\mathcal{L}(t)$ l'espace E_L , de dimension $n+l$, du point $L(q^{\nu}, p_{\lambda}^*)$.

Lorsque, dans un mouvement naturel de $\mathcal{L}(t)$, P décrit une courbe d'extrémités P_1 et P_2 , le point L décrit une courbe d'extrémités L_1 et L_2 , soit :

$$(3) \quad L_j = L(q^{\nu}(t_j), p_{\lambda}^*(t_j)), \quad (j=1, 2),$$

où L_j est la position de L à l'instant t_j .

Dans tout déplacement virtuel de $\mathcal{L}(t)$, compatible avec les liaisons telles qu'elles sont à l'instant t , correspondent dans E_P des δq^{ν} arbitraires et dans E_L les mêmes δq^{ν} et des δp_{λ}^* arbitraires lorsque L varie de L_1 à L_2 :

$$(4) \quad \begin{cases} \delta q_j^{\nu} = \delta q^{\nu}(t_j) = 0, & (j=1, 2), \quad \text{ou} \\ \delta \vec{M}_t(t_j) = 0. \end{cases}$$

Posons :

$$(5) \quad R^*(p_{\lambda}^*, \dot{q}^{\mu}, q^{\mu}, q^{\lambda}; t) \equiv p_{\lambda}^* \dot{q}^{\lambda} - T(\dot{q}^{\nu}, q^{\nu}; t),$$

où le routhien R^* est une fonction, par l'intermédiaire de (2), de p_{λ}^* , $\dot{q}^{\mu}$, q^{λ} et de t .

(La convention sommatoire d'E i n s t e i n est systématiquement adoptée dans ce qui précède comme dans ce qui suit).

Considérons l'expression déjà connue [5]

$$(6) \quad \left\{ \begin{array}{l} \delta T = \frac{d}{dt} \left(\sum_{i \in \mathcal{L}(t)} m_i \vec{V}_i \delta \vec{M}_i \right) - \sum_{i \in \mathcal{L}(t)} m_i \vec{\gamma}_i \delta \vec{M}_i - \\ - \frac{1}{2} \left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^v} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^v} - \frac{\partial \dot{m}_r}{\partial \dot{q}^v} \Lambda_r^* + \frac{\partial \dot{m}_s}{\partial \dot{q}^v} \Lambda_s^* \right) \delta q^v, \\ \text{avec } \vec{\gamma}_i = \vec{V}_i, \end{array} \right.$$

où les $\dot{m}_\alpha \Lambda_\alpha^*$ ($\alpha = M, s$) sont les limites, dans certaines conditions, des dérivées des forces vives des systèmes adjoints des jets dans leur mouvement par rapport à $(\mathcal{X})$; les m_α étant les débits locaux. (Le calcul de ces expressions se fait à l'aide de fonctions associées aux jets d'émission et d'admission).

Dès lors on déduit de (5) et de (6) :

$$(7) \quad \left\{ \begin{array}{l} \delta(p_\lambda^* \dot{q}^\lambda - R^*) \equiv \frac{d}{dt} \left(\sum_{i \in \mathcal{L}(t)} m_i \vec{V}_i \delta \vec{M}_i \right) - \sum_{i \in \mathcal{L}(t)} m_i \vec{\gamma}_i \delta \vec{M}_i - \\ - \frac{1}{2} \left[\left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^v} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^v} \right) - \left(\frac{\partial \dot{m}_r}{\partial \dot{q}^v} \Lambda_r^* - \frac{\partial \dot{m}_s}{\partial \dot{q}^v} \Lambda_s^* \right) \right] \delta q^\lambda. \end{array} \right.$$

Cela étant, considérons l'intégrale,

$$(8) \quad \left\{ \begin{array}{l} \delta_L \mathcal{T} = \int_{t_1}^{t_2} \left\{ \delta(p_\lambda^* \dot{q}^\lambda - R^*) + \frac{1}{2} \left[\left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^v} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^v} \right) - \right. \right. \\ \left. \left. - \left(\frac{\partial \dot{m}_r}{\partial \dot{q}^v} \Lambda_r^* - \frac{\partial \dot{m}_s}{\partial \dot{q}^v} \Lambda_s^* \right) \right] \delta q^v + \delta \mathcal{T} \right\} dt, \\ \text{avec} \\ \delta \mathcal{T} = \sum_{i \in \mathcal{L}(t)} \vec{f}_i \frac{\partial \vec{M}_i}{\partial \dot{q}^v} \delta q = Q_v \delta q^v, \end{array} \right.$$

où $\vec{f}_i$ désigne le système de toutes les forces s'exerçant sur la particule de position $\vec{M}_i(t)$.

Les $\delta \vec{M}_i(t)$ vérifiant (4) aux instants t_1 et t_2 il vient :

$$\int_{t_1}^{t_2} \frac{d}{dt} \left[\sum_{i \in \mathcal{L}(t)} m_i \vec{V}_i \delta \vec{M}_i \right] dt = \left[\sum_{i \in \mathcal{L}(t)} m_i \vec{V}_i \delta \vec{M}_i \right]_{t_1}^{t_2} = 0$$

et (8) s'écrit :

$$\delta_L \mathcal{J}_1^2 = - \int_{t_1}^{t_2} \left[\sum_{i \in \mathcal{L}(t)} (m_i \vec{\gamma}_i - \vec{f}_i) \delta \vec{M}_i \right] dt.$$

Compte tenu du principe du travail virtuel de d'Alembert, cette intégrale est nulle dans tout mouvement voisin du mouvement considéré et donnant les mêmes positions P_1 et P_2 du point figuratif P aux instants t_1 et t_2 .

Il en résulte que l'intégrale (8) est nulle dans tout mouvement voisin du mouvement naturel de $\mathcal{L}(t)$ et pour lequel L varie de L_1 à L_2 . C'est un principe variationnel des systèmes variables qui généralise le principe routhien des systèmes mécaniques classiques (et constants).

2. Les paramètres q^ν ne sont pas indépendants. Nous les supposons liés par des relations de contraintes, en général non intégrables, de la forme

$$(9) \quad \begin{cases} \dot{q}^\alpha = a_\beta^\alpha \dot{q}^\beta + b^\alpha, \\ (\beta = 1, 2, \dots, p; \quad \alpha = p + 1, \dots, n), \end{cases}$$

où a_β^α et b^α dépendent seulement des q^ν et de t .

Nous distinguerons encore, parmi les q^β , l paramètres q^λ ($\lambda = 1, 2, \dots, l$), les $p - l$ autres étant désignés par q^μ ($\mu = l + 1, \dots, p$). Nous associerons au point figuratif $P^*(q^\beta)$ de $\mathcal{L}(t)$ (dans l'espace E_P^* de dimension p) le point $L^*(q^\beta, p_\lambda^*)$ de l'espace E_L^* de dimension $p + l$, et nous définirons le routhien $\bar{R}^*$ par

$$(10) \quad \begin{cases} \bar{R}^*(\bar{p}_\lambda^*, \dot{q}^\mu, q^\nu; t) \equiv \bar{p}_\lambda^* \dot{q}^\lambda - \bar{T}(q^\beta, q^\nu; t), \\ \text{avec} \quad p_\lambda^* = \frac{\partial \bar{T}}{\partial \dot{q}^\lambda}, \\ (\lambda = 1, 2, \dots, n; \quad \beta = 1, 2, \dots, p; \quad \lambda = 1, 2, \dots, l < p), \end{cases}$$

où toute quantité surlignée indique qu'elle est exprimée, à l'instant t , en fonction de q^λ et des seules dérivées des paramètres indépendants q^β .

En se référant à l'expression [2, II G 5 et IV A 10],

$$(11) \quad \begin{cases} \delta \bar{T} = \frac{d}{dt} \left[\sum_{i \in \mathcal{L}(t)} m_i \vec{V}_i \delta \vec{M}_i(t) \right] - \sum_{i \in \mathcal{L}(t)} m_i \vec{\gamma}_i \delta \vec{M}_i(t) + \\ + \left[\frac{\partial \bar{T}}{\partial \dot{q}^\alpha} \left(\frac{\partial \ddot{q}^\alpha}{\partial \dot{q}^\beta} - 2 \bar{a}_\beta^\alpha \right) - \bar{\mathcal{H}}_\beta \right] \delta q^\beta, \\ \text{avec} \\ \bar{\mathcal{H}}_\beta = \frac{1}{2} \left[\left(\bar{m}_r \frac{\partial \bar{\Lambda}_r^*}{\partial \dot{q}^\beta} - \bar{m}_s \frac{\partial \bar{\Lambda}_s^*}{\partial \dot{q}^\beta} \right) - \left(\frac{\partial \bar{m}_r}{\partial \dot{q}^\beta} \bar{\Lambda}_r^* - \frac{\partial \bar{m}_s}{\partial \dot{q}^\beta} \bar{\Lambda}_s^* \right) \right], \end{cases}$$

qui généralise celle qui est relative aux systèmes constants [1], la relation (10) fournit :

$$(12) \quad \left\{ \begin{aligned} \delta(\bar{p}_\lambda^* \dot{q}^\lambda - \bar{R}^*) &= \frac{d}{dt} \left[\sum_{i \in \mathcal{L}(t)} m_i \bar{V}_i \bar{\delta M}_i \right] - \sum_{i \in \mathcal{L}(t)} m_i \bar{\gamma}_i \bar{\delta M}_i + \\ &+ \left[\frac{\partial T}{\partial \dot{q}^\alpha} \left(\frac{\partial \bar{q}^\alpha}{\partial \dot{q}^\beta} - 2\bar{a}_\beta^\alpha \right) - \bar{\mathcal{H}}_\beta \right] \delta q^\beta. \end{aligned} \right.$$

Dès lors on en déduit, en raisonnant comme ci-dessus, que l'intégrale

$$(13) \quad \left\{ \begin{aligned} \delta_{L^*} \mathcal{J}_1^2 &= \int_{t_1}^{t_2} \left\{ \delta(\bar{p}_\lambda^* \dot{q}^\lambda - \bar{R}^*) - \left[\frac{\partial T}{\partial \dot{q}^\alpha} \left(\frac{\partial \bar{q}^\alpha}{\partial \dot{q}^\beta} - 2\bar{a}_\beta^\alpha \right) - \bar{\mathcal{H}}_\beta \right] \delta q^\beta + \delta \bar{\mathcal{C}} \right\} dt, \\ \text{avec } \delta \bar{\mathcal{C}} &= \sum_{i \in \mathcal{L}(t)} \bar{f}_i \bar{\delta M}_i = \bar{Q}_\beta \delta q^\beta, \end{aligned} \right.$$

est nulle dans tout mouvement voisin du mouvement naturel de $\mathcal{L}(t)$ et pour lequel L^* varie de L_1^* à L_2^* .

La nullité de (13), dans les conditions indiquées, traduit le principe variationnel généralisé routhien relatif aux systèmes de particules variables soumis aux contraintes (9).

3. Extension des équations de Routh

1. *Les paramètres q^ν sont indépendants.* Les δq^ν et les δp_λ^* étant alors arbitraires l'on a, pour tout déplacement virtuel de $\mathcal{L}(t)$ respectant t :

$$(14) \quad \delta(p_\lambda^* \dot{q}^\lambda - R^*) = p_\lambda^* \delta \dot{q}^\lambda + \left(\dot{q}^\lambda - \frac{\partial R^*}{\partial p_\lambda^*} \right) \delta p_\lambda^* - \frac{\partial R^*}{\partial q^\lambda} \delta q^\lambda - \frac{\partial R^*}{\partial q^\mu} \delta q^\mu - \frac{\partial R^*}{\partial \dot{q}^\mu} \delta \dot{q}^\mu.$$

Comme les δq^ν satisfont les relations (4), il vient :

$$(15) \quad \left\{ \begin{aligned} \int_{t_1}^{t_2} p_\lambda^* \delta \dot{q}^\lambda dt &= - \int_{t_1}^{t_2} \dot{p}_\lambda^* \delta q^\lambda dt, \\ \int_{t_1}^{t_2} \left(- \frac{\partial R^*}{\partial q^\mu} \delta \dot{q}^\mu \right) dt &= \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial R^*}{\partial \dot{q}^\mu} \right) \right] \delta q^\mu dt. \end{aligned} \right.$$

Tenant compte de (14) et de (15), l'intégrale (8) s'écrit :

$$(16) \quad \left\{ \begin{aligned} \delta_L \mathcal{I}_1^2 &= \int_{t_1}^{t_2} \left\{ \left(\dot{q}^\lambda - \frac{\partial R^*}{\partial p_\lambda} \right) \delta p_\lambda^* + \left[-\dot{p}_\lambda^* - \frac{\partial R^*}{\partial q^\lambda} + \right. \right. \\ &\quad \left. + \frac{1}{2} \left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^\lambda} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^\lambda} \right) - \frac{1}{2} \left(\frac{\partial \dot{m}_r}{\partial \dot{q}^\lambda} \Lambda_r^* - \frac{\partial \dot{m}_s}{\partial \dot{q}^\lambda} \Lambda_s^* \right) + Q_\lambda \right] \delta q^\lambda + \\ &\quad \left. + \left[\frac{d}{dt} \left(\frac{\partial R^*}{\partial \dot{q}^\mu} \right) - \frac{\partial R^*}{\partial q^\mu} + \frac{1}{2} \left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^\mu} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^\mu} \right) - \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \left(\frac{\partial \dot{m}_r}{\partial \dot{q}^\mu} \Lambda_r^* - \frac{\partial \dot{m}_s}{\partial \dot{q}^\mu} \Lambda_s^* \right) + Q_\mu \right] \delta q^\mu dt, \right. \\ &\quad \text{avec} \\ &\quad \left. \delta \mathcal{T} = Q_\lambda \delta q^\lambda + Q_\mu \delta q^\mu, \quad (\lambda = 1, 2, \dots, l; \mu = l + 1, \dots, n). \right\} \end{aligned} \right.$$

Les δq^ν et les δp_λ étant arbitraires, une condition nécessaire et suffisante pour que l'intégrale (16) soit nulle est que les coefficients des δq^μ , δq^λ et des δp_λ^* le soient, d'où les équations du mouvement de $\mathcal{L}(t)$:

$$(17) \quad \left\{ \begin{aligned} \dot{q}^\lambda &= \frac{\partial R^*}{\partial p_\lambda}, \\ \dot{p}_\lambda^* &= -\frac{\partial R^*}{\partial p^\lambda} + \frac{1}{2} \left[\left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^\lambda} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^\lambda} \right) - \left(\frac{\partial \dot{m}_r}{\partial \dot{q}^\lambda} \Lambda_r^* - \frac{\partial \dot{m}_s}{\partial \dot{q}^\lambda} \Lambda_s^* \right) \right] + Q_\lambda, \\ \frac{d}{dt} \left(\frac{\partial R^*}{\partial \dot{q}^\mu} \right) - \frac{\partial R^*}{\partial q^\mu} + \frac{1}{2} \left[\left(\dot{m}_r \frac{\partial \Lambda_r^*}{\partial \dot{q}^\mu} - \dot{m}_s \frac{\partial \Lambda_s^*}{\partial \dot{q}^\mu} \right) - \left(\frac{\partial \dot{m}_r}{\partial \dot{q}^\mu} \Lambda_r^* - \frac{\partial \dot{m}_s}{\partial \dot{q}^\mu} \Lambda_s^* \right) \right] &= -Q_\mu, \\ (\lambda = 1, 2, \dots, l; \mu = l + 1, \dots, n). \end{aligned} \right.$$

Il y a lieu de tenir compte éventuellement de

$$(18) \quad \frac{\partial R^*}{\partial t} = -\frac{\partial T}{\partial t}.$$

Les équations (17₁) et (17₂) ont la forme hamiltonienne généralisée. On y exprimera tous les termes en fonction des p_λ^* , q^λ , q^μ .

Les dernières équations (17₃) ont la forme lagrangienne généralisée.

2. Les paramètres q^ν sont soumis aux contraintes (9). Il s'agit alors d'exprimer que l'intégrale (13) est nulle dans les conditions indiquées.

On a tout d'abord :

$$(19) \quad \left\{ \begin{array}{l} \delta(\bar{p}_\lambda^* \dot{q}^\lambda - \bar{R}^*) \equiv \bar{p}_\lambda^* \delta \dot{q}^\lambda + \left(\dot{q}^\lambda - \frac{\partial \bar{R}^*}{\partial \bar{p}_\lambda^*} \right) \delta \bar{p}_\lambda^* - \\ \quad - \Delta_\lambda \bar{R}^* \delta q^\lambda - \Delta_\mu \bar{R}^* \delta q^\mu - \frac{\partial \bar{R}^*}{\partial \dot{q}^\mu} \delta \dot{q}^\mu, \\ \text{avec} \\ \dot{q}^\alpha = a_\beta^\alpha \dot{q}^\beta + b^\alpha, \quad \Delta_\omega = \frac{\partial}{\partial q^\omega} + a_\omega^\alpha \frac{\partial}{\partial q^\alpha}, \quad \omega = (\lambda, \mu). \end{array} \right.$$

Compte tenu des relations (15) (où l'on remplace p_λ^* par $\bar{p}_\lambda^*$ et R^* par $\bar{R}^*$), le principe variationnel (13) fournit :

$$(20) \quad \left\{ \begin{array}{l} \dot{q}^\lambda = \frac{\partial \bar{R}^*}{\partial \bar{p}_\lambda^*}, \\ \dot{\bar{p}}_\lambda = - \Delta_\lambda \bar{R}^* + \frac{\partial T}{\partial \dot{q}^\alpha} \left(2\bar{a}^\alpha - \frac{\partial \bar{q}^\alpha}{\partial \dot{q}^\lambda} \right) + \\ \quad + \frac{1}{2} \left[\left(\bar{m}_r \frac{\partial \bar{\Lambda}_r^*}{\partial \dot{q}^\lambda} - \bar{m}_s \frac{\partial \bar{\Lambda}_s^*}{\partial \dot{q}^\lambda} \right) - \left(\frac{\partial \bar{m}_r}{\partial \dot{q}^\lambda} \bar{\Lambda}_r^* - \frac{\partial \bar{m}_s}{\partial \dot{q}^\lambda} \bar{\Lambda}_s^* \right) \right], \\ - \bar{Q}_\mu = \frac{d}{dt} \left(\frac{\partial \bar{R}^*}{\partial \dot{q}^\mu} \right) - \Delta_\mu \bar{R}^* + \frac{\partial T}{\partial \dot{q}^\mu} \left(2\bar{a}_\mu^\alpha - \frac{\partial \bar{q}^\alpha}{\partial \dot{q}^\mu} \right) + \\ \quad + \frac{1}{2} \left[\left(\bar{m}_r \frac{\partial \bar{\Lambda}_r^*}{\partial \dot{q}^\mu} - \bar{m}_s \frac{\partial \bar{\Lambda}_s^*}{\partial \dot{q}^\mu} \right) - \left(\frac{\partial \bar{m}_s}{\partial \dot{q}^\mu} \bar{\Lambda}_r^* - \frac{\partial \bar{m}_s}{\partial \dot{q}^\mu} \bar{\Lambda}_s^* \right) \right], \\ \text{avec} \\ \delta \mathcal{T} = \bar{Q}_\lambda \delta q^\lambda + \bar{Q}_\mu \delta q^\mu, \quad \Delta_\omega \bar{R}^* = \frac{\partial \bar{R}^*}{\partial q^\omega} + a_\omega^\alpha \frac{\partial \bar{R}^*}{\partial q^\alpha}. \end{array} \right.$$

On notera que les principes variationnels (8) et (13) ainsi que les équations (17) et (20) qui en résultent fournissent les principes variationnels correspondants relatifs aux systèmes constants. Il suffit seulement d'annuler les débits d'admission $\dot{m}_r$ et d'émission $\dot{m}_s$.

Les équations (17) et (20) seront dites respectivement *équations routhiennes* des mouvements des systèmes variables libres et liés.

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PRINCIPII VARIATIONALE „ROUTHIENE“ ȘI ECUAȚII ROUTHIANE
PENTRU SISTEMELE VARIABLE DE PARTICULE

(Rezumat)

În cadrul sistemelor variabile de particule [2], adică al sistemelor materiale „care schimbă particule cu exteriorul“, se prezintă un principiu variațional denumit routhian, deoarece conduce la obținerea de ecuații analitice ale mișcării sistemelor variabile libere sau legate, care constituie generalizarea ecuațiilor lui Routh relative la sistemele mecanice constante clasice.

НЕКОТОРЫЕ ВОПРОСЫ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ (I)

В. М. СТАРЖИНСКИЙ

0. Введение

Рассматриваются колебания нелинейных автономных систем с произвольным числом $(k+1)$ степеней свободы и близких к консервативным. Последнее означает, что невозмущенная система является консервативной, допускающей интеграл энергии и, более того, предполагается, что характеристическое уравнение ее линейной части имеет по крайней мере одну пару чисто-мнимых корней (во многих задачах механики все его корни оказываются чисто-мнимыми). Тогда невозмущенная система $(2k+2)$ -го порядка может быть преобразована по крайней мере одним способом в систему Ляпунова ([1, § 33]), что и будем предполагать выполненным.

Проводится преобразование исходной (возмущенной) системы, при котором невозмущенная система преобразуется в квазилинейную неавтономную систему $2k$ -го порядка. Для первой и последующей поправок соответствующего (т.е. с теми же начальными условиями) решения возмущенной системы составляются уравнения в вариациях по параметру Пуанкаре [2, гл. II]. Если известно общее решение невозмущенной системы, то интегрирование системы уравнений в вариациях сводится к квадратурам, что и проиллюстрировано на пример.

1. Преобразование возмущенной системы

Рассмотрим (см. введение) систему Ляпунова с демпфированием

$$(1.1) \quad \begin{cases} \frac{d^2 u}{dt^2} + u - U(u, \dot{u}, v_1, \dots, v_k, \dot{v}_1, \dots, \dot{v}_k) = -2\varepsilon F_0(\dot{u}, \dot{v}_1, \dots, \dot{v}_k), \\ \frac{d^2 v_\alpha}{dt^2} + a_{\alpha 1} v_1 + \dots + a_{\alpha k} v_k - \dot{V}_\alpha(u, \dot{u}, v_1, \dots, v_k, \dot{v}_1, \dots, \dot{v}_k) = \\ = -2\varepsilon F_\alpha(\dot{u}, \dot{v}_1, \dots, \dot{v}_k), \quad (\alpha = 1, \dots, k). \end{cases}$$

Здесь $a_{jk} = a_{kj}$ ($j, k = 1, \dots, k$) вещественные постоянные, а $U, V_1, \dots, V_k, F_0, F_1, \dots, F_k$ суть вещественные аналитические функции своих переменных, разложения $F_0, F_1, \dots, F_k$ начинаются с членов не ниже первого порядка, а для $U, V_1, \dots, V_k$ — не ниже второго порядка, наконец, $\varepsilon \geq 0$. Будем предполагать, что невозмущенная система (1.1) т. е. система (1.1) при $\varepsilon = 0$ допускает первый интеграл, который необходимо будет аналитической функцией переменных $u, \dot{u}, v_1, \dots, v_k, \dot{v}_1, \dots, \dot{v}_k$ вида

$$(1.2) \quad H = u^2 + \dot{u}^2 + W(v_1, \dots, v_k, \dot{v}_1, \dots, \dot{v}_k) + \\ + S_3(u, \dot{u}, v_1, \dots, v_k, \dot{v}_1, \dots, \dot{v}_k) = \mu^2, \quad (\mu > 0),$$

где W — квадратичная форма, а S_3 — совокупность членов не ниже третьего порядка. Для выделенных сил сопротивления $F_0, F_1, \dots, F_k$ будем предполагать, что их работа на любом возможном (совпадающем в данном случае с одним из действительных) перемещений отрицательна:

$$(1.3) \quad - \sum_{j=0}^k F_j(\dot{v}_0, \dot{v}_1, \dots, \dot{v}_k) \dot{v}_j < 0, \quad (\dot{v}_0 \equiv \dot{u}).$$

В простейшем нелинейном случае, когда $F_j = F(v_j)$ ($j = 0, 1, \dots, k$; $\dot{v}_0 \equiv \dot{u}$), условие (1.3) означает, что $\alpha F(\alpha) > 0$, ($\alpha \neq 0$). В линейном случае условие (1.3) означает, что диссипация полная.

Подстановка Ляпунова

$$(1.4) \quad \begin{cases} u = \rho \sin \theta, & \dot{u} = \rho \cos \theta, & (\rho \geq 0, \theta \geq \theta_0), \\ v_k = \rho z_k, & \dot{v}_k = \rho z_{k+\kappa}, & (\kappa = 1, \dots, k), \end{cases}$$

приведет систему (1.1) и первый интеграл (1.2) невозмущенной системы к виду

$$(1.5) \quad \begin{cases} \frac{d\rho}{dt} = U(\rho \sin \theta, \rho \cos \theta, \rho \mathbf{z}) \cos \theta - 2\varepsilon F_0(\rho \cos \theta, \rho \mathbf{z}^{(2)}) \cos \theta, \\ \frac{d\theta}{dt} = 1 - \frac{1}{\rho} U(\rho \sin \theta, \rho \cos \theta, \rho \mathbf{z}) \sin \theta + 2\varepsilon \frac{1}{\rho} F_0(\rho \cos \theta, \rho \mathbf{z}^{(2)}) \sin \theta, \\ \frac{dz_x}{dt} = z_{k+\kappa} - \frac{1}{\rho} U(\rho \sin \theta, \rho \cos \theta, \rho \mathbf{z}) \cos \theta + \\ + 2\varepsilon \frac{1}{\rho} z_{\kappa} F_0(\rho \cos \theta, \rho \mathbf{z}^{(2)}) \cos \theta, \end{cases}$$

$$(1.6) \quad \begin{cases} \frac{dz_{k+\kappa}}{dt} = -a_{\kappa 1} z_1 - \dots - a_{\kappa k} z_k - \frac{1}{\rho} z_{k+\kappa} U(\rho \sin \theta, \rho \cos \theta, \rho \mathbf{z}) \cos \theta + \\ + \frac{1}{\rho} V_{\kappa}(\rho \sin \theta, \rho \cos \theta, \rho \mathbf{z}) + 2\varepsilon \frac{1}{\rho} z_{k+\kappa} F_0(\rho \cos \theta, \rho \mathbf{z}^{(2)}) \cos \theta - \\ - 2\varepsilon F_{\kappa}(\rho \cos \theta, \rho \mathbf{z}^{(2)}), \quad (\kappa = 1, \dots, k); \\ \rho^2 [1 + W(\mathbf{z}) + \rho S(\rho, \theta, \mathbf{z})] = \mu^2. \end{cases}$$

Здесь $\mathbf{z}$ и $\mathbf{z}^{(2)}$ суть векторы с компонентами соответственно $z_1, \dots, z_{2k}$ и $z_{k+1}, \dots, z_{2k}$, а $S = \rho^{-3} S_3$.

Предположим для дальнейшего, что в параллелепипеде $|z| \leq c$, $0 \leq \rho \leq r$ и при любом $\theta \geq \theta_0$ для всех $\varepsilon \in [0, \varepsilon_0]$ выполнено неравенство

$$(1.7) \quad \left| \frac{1}{\rho} U(\rho \sin \theta, \rho \cos \theta, \rho \mathbf{z}) \sin \theta - 2\varepsilon \frac{1}{\rho} F_0(\rho \cos \theta, \rho \mathbf{z}^{(2)}) \sin \theta \right| < 1.$$

При условии (1.7) разделим второе и последующие уравнения системы (1.5) на первое:

$$(1.8) \quad \begin{cases} \frac{d\rho}{d\theta} = \frac{U \cos \theta - 2\varepsilon F_0 \cos \theta}{1 - \rho^{-1} U \sin \theta + 2\varepsilon \rho^{-1} F_0 \sin \theta}, \\ \frac{dz_x}{d\theta} = \frac{z_{k+x} - \rho^{-1} z_x U \cos \theta + 2\varepsilon \rho^{-1} z_x F_0 \cos \theta}{1 - \rho^{-1} U \sin \theta + 2\varepsilon \rho^{-1} F_0 \sin \theta}, \\ \frac{dz_{k+x}}{d\theta} = (1 - \rho^{-1} U \sin \theta + 2\varepsilon \rho^{-1} F_0 \sin \theta)^{-1} (-a_{x1} z_1 - \dots - a_{xk} z_k - \\ - \rho^{-1} z_{k+x} U \cos \theta + \rho^{-1} V_x + 2\varepsilon \rho^{-1} z_{k+x} F_0 \cos \theta - 2\varepsilon \rho^{-1} F_x), \\ (\chi = 1, \dots, k). \end{cases}$$

В [3]...[5] невозмущенная система (1.8) (т.е. при $\varepsilon = 0$) за счет использования интеграла (1.6) сведена к квазилинейной неавтономной системе $2k$ -го порядка; ее решение определяется методами малого параметра для достаточно малых положительных значений μ в (1.6).

2. Уравнения в вариациях Пуанкаре и их решение

Запишем систему уравнений (1.8) в виде векторного уравнения

$$(2.1) \quad \frac{d\mathbf{x}}{d\theta} = \mathbf{f}(\theta, \mathbf{x}; \varepsilon),$$

где $\mathbf{x}$ — вектор с компонентами $\rho, z_1, \dots, z_{2k}$, а $\mathbf{f}$ — вектор-функция, составленная из правых частей системы (1.8), аналитическая по $\mathbf{x}$ и ε в области определения (1.7), а коэффициенты степенных рядов по $\rho, z_1, \dots, z_{2k}$ суть периодические функции θ периода 2π . Допустим нам известно порождающее решение $x_0(\theta)$ невозмущенного (т.е. при $\varepsilon = 0$) уравнения (2.1)

$$(2.2) \quad \frac{dx_0}{d\theta} = \mathbf{f}(\theta, \mathbf{x}_0; 0).$$

Основываясь на теореме Пуанкаре [2, гл. II] будем искать решение уравнения (2.1) для достаточно малых положительных значений ε в виде

$$(2.3) \quad \mathbf{x} = \sum_{m=0}^{\infty} \varepsilon^m \mathbf{x}_m(\theta).$$

Вычитая из уравнения (2.1) тождество (2.2) и используя формулу Тейлора для функции многих переменных, получим в силу (2.3)

$$\sum_{m=1}^{\infty} \varepsilon^m \frac{d\mathbf{x}_m}{d\theta} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\partial}{\partial \mathbf{x}} \sum_{m=1}^{\infty} \varepsilon^m \mathbf{x}_m + \frac{\partial}{\partial \varepsilon} \varepsilon \right)^n \mathbf{f}(\theta, \mathbf{x}_0; 0).$$

Приравнявая коэффициенты при одинаковых степенях ε получим последовательность весторных дифференциальных уравнений (уравнения в вариациях по параметру Пуанкаре)

$$(2.4) \quad \left\{ \begin{aligned} \frac{d\mathbf{x}_1}{d\theta} &= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_0 \mathbf{x}_1 + \left(\frac{\partial \mathbf{f}}{\partial \varepsilon} \right)_0, \\ \frac{d\mathbf{x}_2}{d\theta} &= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_0 \mathbf{x}_2 + \frac{1}{2} \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \right)_0 \mathbf{x}_1 \mathbf{x}_1 + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon} \right)_0 \mathbf{x}_1 + \frac{1}{2} \left(\frac{\partial^2 \mathbf{f}}{\partial \varepsilon^2} \right)_0, \\ \frac{d\mathbf{x}_3}{d\theta} &= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_0 \mathbf{x}_3 + \frac{1}{2} \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \right)_0 (\mathbf{x}_1 \mathbf{x}_2 + \mathbf{x}_2 \mathbf{x}_1) + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon} \right)_0 \mathbf{x}_2 + \\ &\quad + \frac{1}{6} \left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3} \right)_0 \mathbf{x}_1 \mathbf{x}_1 \mathbf{x}_1 + \frac{1}{2} \left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^2 \partial \varepsilon} \right)_0 \mathbf{x}_1 \mathbf{x}_1 + \frac{1}{2} \left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon^2} \right)_0 \mathbf{x}_1 + \frac{1}{6} \left(\frac{\partial^3 \mathbf{f}}{\partial \varepsilon^3} \right)_0, \\ \frac{d\mathbf{x}_m}{d\theta} &= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_0 \mathbf{x}_m + \frac{1}{2} \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \right)_0 \sum_{\alpha_1 + \alpha_2 = m} \mathbf{x}_{\alpha_1} \mathbf{x}_{\alpha_2} + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon} \right)_0 \mathbf{x}_{m-1} + \\ &\quad + \frac{1}{6} \left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3} \right)_0 \sum_{\alpha_1 + \alpha_2 + \alpha_3 = m} \mathbf{x}_{\alpha_1} \mathbf{x}_{\alpha_2} \mathbf{x}_{\alpha_3} + \frac{1}{2} \left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^2 \partial \varepsilon} \right)_0 \sum_{\alpha_1 + \alpha_2 = m-1} \mathbf{x}_{\alpha_1} \mathbf{x}_{\alpha_2} + \\ &\quad + \frac{1}{2} \left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon^2} \right)_0 \mathbf{x}_{m-2} + \dots + \frac{1}{n!} \left(\frac{\partial^n \mathbf{f}}{\partial \mathbf{x}^n} \right)_0 \sum_{\alpha_1 + \dots + \alpha_n = m} \mathbf{x}_{\alpha_1} \dots \mathbf{x}_{\alpha_n} + \\ &\quad + \frac{1}{(n-1)!} \left(\frac{\partial^n \mathbf{f}}{\partial \mathbf{x}^{n-1} \partial \varepsilon} \right)_0 \sum_{\alpha_1 + \dots + \alpha_{n-1} = m-1} \mathbf{x}_{\alpha_1} \dots \mathbf{x}_{\alpha_{n-1}} + \\ &\quad + \dots + \frac{1}{(n-l)! l!} \left(\frac{\partial^n \mathbf{f}}{\partial \mathbf{x}^{n-l} \partial \varepsilon^l} \right)_0 \sum_{\alpha_1 + \dots + \alpha_{n-l} = m-l} \mathbf{x}_{\alpha_1} \dots \mathbf{x}_{\alpha_{n-l}} + \dots + \\ &\quad + \frac{1}{(n-1)!} \left(\frac{\partial^n \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon^{n-1}} \right)_0 \mathbf{x}_{m-n+1} + \dots + \frac{1}{m!} \left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^m} \right)_0 \mathbf{x}_1^m + \\ &\quad + \frac{1}{(m-1)!} \left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^{m-1} \partial \varepsilon} \right)_0 \mathbf{x}_1^{m-1} + \dots + \frac{1}{(m-l)! l!} \left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^{m-l} \partial \varepsilon^l} \right)_0 \mathbf{x}_1^{m-l} + \dots + \\ &\quad + \frac{1}{(m-1)!} \left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x} \partial \varepsilon^{m-1}} \right)_0 \mathbf{x}_1 + \frac{1}{m!} \left(\frac{\partial^m \mathbf{f}}{\partial \varepsilon^m} \right)_0. \end{aligned} \right.$$

Здесь индекс нуль означает, что значения частных производных вычислены при $\mathbf{x} = \mathbf{x}_0(0)$ и $\varepsilon = 0$; $\alpha_1, \alpha_2, \dots$ суть натуральные числа. Если векторы $\mathbf{x}$ и $\mathbf{f}$ мы запишем как

$$\mathbf{x} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_p \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_p \end{pmatrix},$$

то производная вектора по вектору $\partial \mathbf{f} / \partial \mathbf{x}$ есть матрица

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left\| \frac{\partial \varphi_j}{\partial \xi_h} \right\|_1^p.$$

Последующие члены в уравнениях (2.4) также нужно трактовать в операторном смысле, так, например,

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \mathbf{x}_1 \mathbf{x}_1 = \frac{\partial \left[\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{x}_1 \right]}{\partial \mathbf{x}} \mathbf{x}_1.$$

Этим и объясняется, что мы избегали выписывать степени $\mathbf{x}$ и различали, например, члены с $\mathbf{x}_1 \mathbf{x}_2$ и $\mathbf{x}_2 \mathbf{x}_1$ из-за некоммутативности, вообще говоря, оператора. Если ε входит линейно, т.е. (2.1) имеет вид

$$(2.5) \quad \frac{d\mathbf{x}}{d\theta} = \mathbf{g}(\theta, \mathbf{x}) + \varepsilon \mathbf{h}(\theta, \mathbf{x}),$$

то уравнения в вариациях по параметру Пуанкаре могут быть записаны в виде

$$(2.6) \quad \begin{aligned} \frac{d\mathbf{x}_1}{d\theta} &= \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right)_0 \mathbf{x}_1 + \mathbf{h}(\theta, \mathbf{x}_0), \\ \frac{d\mathbf{x}_m}{d\theta} &= \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}} \right)_0 \mathbf{x}_m + \sum_{v=2}^m \left[\frac{1}{v!} \left(\frac{\partial^v \mathbf{g}}{\partial \mathbf{x}^v} \right)_0 \sum_{\alpha_1 + \dots + \alpha_v = m} \mathbf{x}_{\alpha_1} \dots \mathbf{x}_{\alpha_v} + \right. \\ &\quad \left. + \frac{1}{(v-1)!} \left(\frac{\partial^{v-1} \mathbf{h}}{\partial \mathbf{x}^{v-1}} \right)_0 \sum_{\alpha_1 + \dots + \alpha_{v-1} = m-1} \mathbf{x}_{\alpha_1} \dots \mathbf{x}_{\alpha_{v-1}} \right], \quad (m > 1). \end{aligned}$$

Уравнения (2.4) и (2.6) интегрируются последовательно непосредственно только лишь в скалярном случае. Но даже в этом случае (а тем более и в векторном случае) полезно напомнить положение Пуанкаре [2, гл. 11], что если известно общее решение невозмущенного уравнения (2.2), то интегрирование систем уравнений (2.4) любого порядка сводится к квадратурам. Действительно, пусть

$$\mathbf{x}_0 = \mathbf{x}_0(\theta; \mathbf{a}),$$



где $\mathbf{x}_0$, $\mathbf{a}$ — векторы размерности p , есть общее решение уравнения (2.1) при $\varepsilon = 0$, т.е.

$$\frac{d\mathbf{x}_0(\theta; \mathbf{a})}{d\theta} \equiv \mathbf{f}(\theta, \mathbf{x}_0(\theta; \mathbf{a}); 0).$$

Дифференцируя это тождество по $\mathbf{a}$, получим

$$\frac{d\left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)}{d\theta} \equiv \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right).$$

Отсюда следует, что $\partial \mathbf{x}_0 / \partial \mathbf{a}$ есть фундаментальная матрица каждой из однородных систем дифференциальных уравнений, соответствующих (2.4). Тогда решение первой из систем (2.4) с нулевыми начальными условиями $\mathbf{x}_1(\theta_0) = 0$ (ибо $\mathbf{x}_0(\theta_0; \mathbf{a}) = \mathbf{x}^{(0)}$, $\mathbf{x}_\nu(\theta_0) = 0$, ($\nu = 1, 2, \dots$)) запишется в виде

$$(2.7) \quad \mathbf{x}_1 = \frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} \int_{\theta_0}^{\theta} \left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)^{-1} \left(\frac{\partial \mathbf{f}}{\partial \varepsilon}\right)_0 d\theta,$$

где $\partial \mathbf{x}_0 / \partial \mathbf{a}$ — неособая матрица в силу того, что решение $\mathbf{x}_0 = \mathbf{x}_0(\theta; \mathbf{a})$ — общее. Для $\mathbf{x}_m$ ($m > 1$) получатся формулы, аналогичные (2.7) с заменой под знаком интеграла $(\partial \mathbf{f} / \partial \varepsilon)_0$ на неоднородную часть соответствующей системы (2.4).

3. Механическая система с одной степенью свободы

Пусть названная система подвержена действию восстанавливающей силы и силы сопротивления, предполагающимися аналитическими функциями соответственно координаты и скорости. Интегрирование проводится в предположении достаточно малых по сравнению с единицей а) корня квадратного из приведенной начальной энергии системы и б) коэффициента демпфирования. Уравнение движения системы при подходящем выборе масштаба времени t может быть записано в виде (1.1)

$$(3.1) \quad \ddot{u} + u - U(u) = -2\varepsilon F(\dot{u}),$$

где $F(\dot{u})$ и $U(u)$ — аналитические функции во всей области изменения переменных, разложения которых начинаются соответственно с членов не ниже первой и второй степени, кроме того работа силы сопротивления предполагается положительной на любом перемещении: $F(\dot{u})\dot{u} > 0$ ($\dot{u} \neq 0$), что, в частности, выполнено для нечетной функции $F(\dot{u})$. При $\varepsilon = 0$ невозмущенное уравнение (3.1) допускает интеграл энергии

$$(3.2) \quad 2H = u^2 + \dot{u}^2 + 2V(u) = \mu^2, \quad \left(V(u) = - \int_0^u U(u) du, \mu > 0 \right).$$

Запишем уравнение (3.1) в виде системы

$$\frac{du}{dt} = \dot{u}, \quad \frac{d\dot{u}}{dt} = -u + U(u) - 2\varepsilon F(\dot{u})$$

и применим подстановку Ляпунова

$$(3.3) \quad u = \rho \sin \theta, \quad \dot{u} = \rho \cos \theta, \quad (\rho \geq 0, \quad \theta \geq \theta_0).$$

Разрешая относительно $\dot{\rho}$ и $\dot{\theta}$, получим

$$(3.4) \quad \begin{cases} \frac{d\rho}{dt} = U(\rho \sin \theta) \cos \theta - 2\varepsilon F(\rho \cos \theta) \cos \theta, \\ \frac{d\theta}{dt} = 1 - \frac{1}{\rho} U(\rho \sin \theta) \sin \theta + 2\varepsilon \frac{1}{\rho} F(\rho \cos \theta) \sin \theta. \end{cases}$$

Предположим для дальнейшего, что для всех ε и ρ в прямоугольнике $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \rho \leq r$ и при любом $\theta \geq \theta_0$ выполнено неравенство (1.7). Неравенство (1.7) вместе с формулами (3.3) определит допустимую область рассмотрения в плоскости $u, \dot{u}$. При условии (1.7) разделим первое уравнение (3.4) на второе:

$$(3.5) \quad \frac{d\rho}{d\theta} = \frac{U(\rho \sin \theta) \cos \theta - 2\varepsilon F(\rho \cos \theta) \cos \theta}{1 - \frac{1}{\rho} U(\rho \sin \theta) \sin \theta + 2\varepsilon \frac{1}{\rho} F(\rho \cos \theta) \sin \theta}.$$

Интеграл уравнения (3.5) при $\varepsilon = 0$ найдем подставив (3.3) в интеграл энергии (3.2):

$$(3.6) \quad \rho^2 + 2V(\rho \sin \theta) = \mu^2,$$

при этом разложение второго слагаемого начинается с членов не ниже третьей степени ρ . Поскольку $\rho \geq 0$, то уравнение (3.6) допускает единственное решение

$$(3.7) \quad \rho = \rho_0(\theta; \mu) = \mu - \frac{1}{\mu} V(\mu \sin \theta) + \frac{1}{\mu^2} \left(\frac{\partial V}{\partial \rho} \right)_{\rho=\mu} V(\mu \sin \theta) \sin \theta - \frac{1}{2} \frac{1}{\mu^3} \left[V(\mu \sin \theta) \right]^2 + O(\mu^4)$$

для достаточно малых положительных μ (ниже для примера будет определен радиус сходимости ряда (3.7). Обозначим правую часть уравнения (3.5) через $f(\theta, \rho; \varepsilon)$. Из (3.7) и (3.5) найдем

$$(3.8) \quad \frac{\partial \rho_0}{\partial \mu} = 1 - \frac{\partial}{\partial \mu} \left[\frac{1}{\mu} V(\mu \sin \theta) \right] + O(\mu^2),$$

$$(3.9) \quad \left(\frac{\partial F}{\partial \varepsilon} \right)_0 = -2 \left[1 - \frac{1}{\rho_0} U(\rho_0 \sin \theta) \sin \theta \right]^{-1} F(\rho_0 \cos \theta) \cos \theta \times \\ \times \left\{ 1 + \frac{1}{\rho_0} U(\rho_0 \sin \theta) \sin \theta \left[1 - \frac{1}{\rho_0} U(\rho_0 \sin \theta) \sin \theta \right]^{-1} \right\}.$$

Теперь из (2.7), (3.7) и (3.3) получим

$$(3.10) \quad u = [\rho_0(\theta; \mu) + \varepsilon \rho_1(\theta; \mu) + O(\varepsilon^2)] \sin \theta$$

и после определения $\theta = \theta(t; \theta_0, \mu, \varepsilon)$ из второго уравнения (3.4) найдем общее решение исходного уравнения (3.1) (с произвольными постоянными θ_0 и μ) для достаточно малых ε , определяемых условием (1.7) и условием цитированной в §2 теоремы Пуанкаре.

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UNELE PROBLEME ALE TEORIEI OSCILAȚIILOR NELINIARE (I)

(Rezumat)

Se consideră oscilațiile unui sistem neliniar autonom cu număr arbitrar de grade de libertate, $k + 1$, care în regim neperturbat este conservativ. Se propune o transformare a sistemului perturbat, în urma căreia sistemul neperturbat trece într-un sistem neautonom cuasiliniat de ordinul $2k$. Pentru prima și a doua aproximație a soluției sistemului perturbat se stabilesc ecuațiile în variații de tip Poincaré. Când se cunoaște soluția generală a sistemului neperturbat, integrarea sistemului de ecuații în variații se reduce la cuadraturi.

UNE RÉOLUTION EXACTE DES ÉQUATIONS DE VERSTRAETEN ET ROMAIN

PAR

VIOREL MURGESCU et VICTOR CIUBOTARU

1. Introduction

Dans un *Essai d'interprétation visco-élastique du comportement des sols sous chargements répétés* J. Verstraeten et J. Romain ont obtenu à l'aide de la transformation de Carson les formules donnant le déplacement vertical maximal $e_{M,n}$ et le déplacement vertical minimal $e_{m,n}$ au cours du n^e cycle de chargement, appliqué aux massifs semi-infinis [1, pp. 88—89].

Dans le cas d'une charge positive en créneaux ces formules ont la forme ¹⁾

$$(1.1) \quad e_{M,n} = C_1 + \sum_{j=2}^{k-1} C_j \frac{1 - e^{-T'/\tau_j}}{1 - e^{-T/\tau_j}} (1 - e^{-nT/\tau_j}) + C'_k T'^n$$

et

$$(1.2) \quad e_{m,n} = \sum_{j=2}^{k-1} C_j \frac{1 - e^{-T'/\tau_j}}{1 - e^{-T/\tau_j}} (1 - e^{-nT/\tau_j}) e^{-(T-T')/\tau_j} + C'_k T'^n,$$

où on a fait les notations suivantes : p_0 — charge constante uniformément répartie sur un cercle de rayon a ; ν — coefficient de Poisson ; $K = 2(1 - \nu^2)$; E_j , ($j = 1, 2, \dots, k-1$) — modules d'élasticité des éléments du modèle de Burgers généralisé (fig. 1) ; λ_j , ($j = 2, 3, \dots, k$) — coefficients de viscosité de Trouton des éléments du modèle ;

$$(1.3) \quad \tau_j = \frac{\lambda_j}{E_j}, \quad (j = 2, 3, \dots, k-1),$$

temps de retardation des cellules composant le modèle ;

1) On a utilisé les notations du [1].

$$(1.4) \quad C_j = \frac{K a p_0}{E_j}, \quad (j = 1, 2, \dots, k-1);$$

$$(1.5) \quad C'_k = \frac{K a p_0}{\lambda_k};$$

T — période ; T' — durée d'une mise en charge ; $e_{M,n}$ — déplacement vertical maximal au cours du n^e cycle de chargement ; $e_{m,n}$ — déplacement vertical minimal au cours du n^e cycle de chargement.

Aux formules (1.1) et (1.2) correspond le modèle de Burgers généralisé pour un système viscoélastique linéaire (fig. 1). Le premier terme de (1.1) représente la contribution de l'élasticité instantanée, le deuxième celle de l'élasticité retardée et le troisième, celle de la viscosité pure.

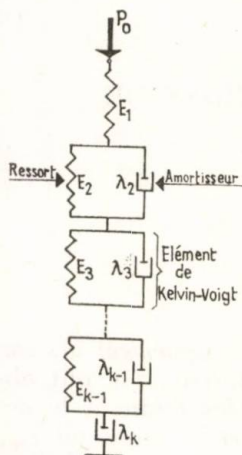


Fig. 1

En considérant les formules (1.1) et (1.2) comme un système d'équations dont les inconnues sont C_1 , C_j , τ_j et C'_k , les grandeurs $e_{M,n}$ et $e_{m,n}$ étant connues par expérience pour une suite $\{n_i\}$ de cycles, on poursuit dans ce qui suit de fournir une solution du système, utile pour la détermination de quelques propriétés caractéristiques de divers sols utilisés dans la construction des routes.

Comme il est impossible d'obtenir des solutions approximatives par des développements en séries potentielles à cause de la lente convergence des séries alternées qui apparaissent dans les calculs, dans ce qui suit on a cherché à résoudre le système (1.1), (1.2) pour les cas des modèles de Burgers contenant une, deux ou trois cellules par une voie élémentaire.

La simplicité de la forme des résultats et la méthode d'élimination conçue à ce but nous ont fait croire qu'il serait utile de les faire présenter dans cette Note.

2. La résolution du système pour le modèle de Burgers à une cellule

Le système (1.1), (1.2) ayant dans ce cas la forme

$$(2.1) \quad e_{M,n} = C_1 + C_2 \frac{1 - e^{-T'/\tau}}{1 - e^{-T/\tau}} (1 - e^{-nT/\tau}) + C'_3 T' n,$$

$$(2.2) \quad e_{m,n} = C_2 \frac{1 - e^{-T'/\tau}}{1 - e^{-T/\tau}} (1 - e^{-nT/\tau}) e^{-(T-T')/\tau} + C'_3 T' n,$$

en supposant dans (2.1) les valeurs de C_2 , τ et C'_3 connues, on en obtient la valeur C_1^0 de C_1 . Alors, la relation (1.4) devient

$$(2.3) \quad C_1^0 = \frac{K a p_0}{E_1},$$

d'où il en résulte

$$(2.4) \quad E_1 = \frac{K a p_0}{C_1^0}.$$

Donc, il reste seulement l'équation (2.2) comme essentielle pour obtenir les expressions des inconnues

$$(2.5) \quad C_2 = \frac{K a p_0}{E_2}, \quad C_3' = \frac{K a p_0}{\lambda_3}, \quad \tau = \frac{\lambda_2}{E_2}.$$

Dans ce but on considère une terme quelconque (n_1, n_2, n_3) à termes approchés, $(n_1 < n_2 < n_3)$, de la suite $\{n_p\}$ des cycles de chargements pour lesquels on suppose les variables (2.5) pratiquement invariables. Alors, e_{m,n_j} étant les déplacements minimaux correspondants, de (2.2) on obtient le système de trois équations à trois inconnues

$$(2.6) \quad C_2 \frac{1 - e^{-T'/\tau}}{1 - e^{-T/\tau}} (1 - e^{-n_j T/\tau}) e^{-(T-T')/\tau} + C_3' T' n_j = e_{m,n_j}, \quad (j = 1, 2, 3).$$

En éliminant C_3' de deux paires de ces équations, on obtient le système réduit

$$(2.7) \quad \begin{aligned} C_2 \frac{1 - e^{-T'/\tau}}{1 - e^{-T/\tau}} [(n_1 - n_2) + (n_2 e^{-n_1 T/\tau} - n_1 e^{-n_2 T/\tau})] e^{-(T-T')/\tau} &= \\ &= n_1 e_{m,n_2} - n_2 e_{m,n_1}, \\ C_2 \frac{1 - e^{-T'/\tau}}{1 - e^{-T/\tau}} [(n_2 - n_3) + (n_3 e^{-n_2 T/\tau} - n_2 e^{-n_3 T/\tau})] e^{-(T-T')/\tau} &= \\ &= n_2 e_{m,n_3} - n_3 e_{m,n_2}, \end{aligned}$$

d'où, par division terme à terme, il résulte l'équation

$$(2.8) \quad \frac{(n_1 - n_2) + (n_2 e^{-n_1 T/\tau} - n_1 e^{-n_2 T/\tau})}{(n_2 - n_3) + (n_3 e^{-n_2 T/\tau} - n_2 e^{-n_3 T/\tau})} = \frac{n_1 e_{m,n_2} - n_2 e_{m,n_1}}{n_2 e_{m,n_3} - n_3 e_{m,n_2}},$$

contenant une seule inconnue, τ .

En faisant les notations

$$(2.9) \quad a_{12} = n_1 e_{m,n_2} - n_2 e_{m,n_1}, \quad a_{23} = n_2 e_{m,n_3} - n_3 e_{m,n_2}$$

et la substitution

$$(2.10) \quad z = e^{-T/\tau},$$

l'équation (2.8) devient

$$(2.8') \quad a_{12} n_2 (z^{n_3} - 1) - (a_{12} n_3 + a_{23} n_1) (z^{n_2} - 1) + a_{23} n_2 (z^{n_1} - 1) = 0.$$

Vu que $T > 0$ et $z > 0$, la solution $z = 1$ apparaissant comme une solution limite pour $\tau = \infty$, le problème de déterminer d'autres solutions de cette équation, pour des ternes (n_1, n_2, n_3) quelconques, reste ouvert. Malgré cela, il existe au moins une possibilité de résoudre l'équation (2.8') en prenant

$$(2.11) \quad n_1 = \mu, \quad n_2 = 2\mu, \quad n_3 = 3\mu, \quad (\mu \in N),$$

et en notant

$$(2.12) \quad z^\mu = w.$$

Alors, l'équation (2.8') devient d'abord

$$a_{12}n_2(w^3 - 1) - (a_{12}n_3 + a_{23}n_1)(w^2 - 1) + a_{23}n_2(w - 1) = 0,$$

et en éliminant le facteur $w - 1 \neq 0$, on obtient l'équation de deuxième degré

$$a_{12}n_2w^2 + (a_{12}n_2 - a_{12}n_3 - a_{23}n_1)w + (a_{12}n_2 + a_{23}n_2 - a_{12}n_3 - a_{23}n_1) = 0.$$

Compte tenu de (2.9) et (2.11), cette équation prend la forme finale

$$(2.13) \quad (e_{m,2\mu} - 2e_{m,\mu})w^2 + (e_{m,\mu} + e_{m,2\mu} - e_{m,3\mu})w + (e_{m,\mu} - 2e_{m,2\mu} + e_{m,3\mu}) = 0,$$

où $e_{m,\mu}$, $e_{m,2\mu}$ et $e_{m,3\mu}$ sont connus et

$$(2.14) \quad w = e^{-\mu T/\tau},$$

conformément aux notations (2.12) et (2.10). Mais $w = 1$ étant une solution de l'équation (2.13), il en résulte que la seule solution acceptable pour ce problème est donnée par

$$(2.15) \quad w_0 = \frac{2e_{m,2\mu} - (e_{m,\mu} + e_{m,3\mu})}{2e_{m,\mu} - e_{m,2\mu}}, \quad (\mu \in N),$$

où on a toujours $w_0 > 0$ d'après l'évaluation de quelques suites des données expérimentales.

En notant par τ_0 la valeur de τ qui correspond à w_0 , on tire de (2.14) la formule

$$(2.16) \quad \tau_0 = -\frac{\mu T}{\ln w_0}.$$

Maintenant, on peut déterminer C_2 en considérant l'une des équations (2.7), par exemple la première, ou on remplace n_1 par μ et n_2 par 2μ . On obtient finalement

$$(2.17) \quad C_2^0 = \frac{2e_{m,\mu} - e_{m,2\mu}}{(1 - w_0)^2} \frac{1 - w_0^{1/\mu}}{1 - w_0^{T'/(\mu T)}} w_0^{(T' - T)/(\mu T)}.$$

Alors, de (2.5) il résulte

$$(2.18) \quad E_2^0 = \frac{K \alpha p_0}{C_2^0}$$

et

$$(2.19) \quad \lambda_2^0 = E_2^0 \tau_0 = - \frac{\mu T E_2^0}{\ln w_0}.$$

En faisant en (2.6), $n_1 = \mu$, on obtient pour C_3' l'expression

$$C_3' = \frac{e_{m,2\mu} - (1 + w_0) e_{m,\mu}}{\mu T' (1 - w_0)},$$

d'où il résulte — compte tenu de (2.5) —,

$$(2.20) \quad \lambda_3^0 = \frac{K \alpha p_0 \mu T' (1 - w_0)}{e_{m,2\mu} - (1 + w_0) e_{m,\mu}}.$$

En conclusion, tous les éléments du modèle de Burgers à une cellule, résultant des formules de Verstraeten et Romain, peuvent être calculés avec les formules (2.4), (2.15), (2.16), (2.18) ... (2.20).

3. La résolution du système pour le modèle de Burgers à deux cellules

Les inconvénients signalés dans le cadre du paragraphe précédent concernant la résolution du système (1.1), (1.2) sont agrandis cette fois parce que le nombre des inconnues a augmenté de quatre à six. Par conséquent, on cherchera à résoudre le système (1.1), (1.2) qui correspond au modèle à deux cellules non pas d'une manière tout à fait générale, mais pour un ensemble de données expérimentales d'indices convenablement choisis,

$$(3.1) \quad n_k = k\mu, \quad (k = 1, 2, \dots, 5; \mu \in N),$$

pour lesquelles on supposera de nouveau les inconnues restant pratiquement invariables.

Soit donc

$$(3.2) \quad e_{M,n_1} = C_1 + \sum_{k=2}^3 C_k \frac{1 - e^{-T'/\tau_k}}{1 - e^{-T/\tau_k}} (1 - e^{-n_1 T/\tau_k}) + C_4 T' n_1$$

et

$$(3.3) \quad e_{m,n_k} = \sum_{j=2}^3 C_j \frac{1 - e^{-T'/\tau_j}}{1 - e^{-T/\tau_j}} (1 - e^{-n_k T/\tau_j}) e^{-(T-T')/\tau_j} + C'_4 T' n_k, \\ (\quad = 1, \dots, 5),$$

le système de six équations à résoudre, où les inconnues sont données par les formules

$$(3.4) \quad E_j = \frac{Kap_0}{C_j}, \quad \tau_j = \frac{\lambda_j}{E_j}, \quad (j = 2, 3), \quad \lambda_4 = \frac{Kap_0}{C'_4},$$

dont la signification a été exposé dans le premier paragraphe.

En supposant dans (3.2) les valeurs de C_k , τ_k et C'_4 connues et en notant la valeur correspondante de C_1 par C_1^0 , on obtient directement de (3.2) et (1.4),

$$(3.5) \quad E_1 = \frac{Kap_0}{C_1^0}.$$

En notant

$$(3.6) \quad e^{-T/\tau_2} = z_2, \quad e^{-T/\tau_3} = z_3$$

et

$$(3.7) \quad C_2 \frac{1 - z_2^{T'/T}}{1 - z_2} z_2^{1-T'/T} (z_2^\mu - 1)^2 = \alpha_2, \quad C_3 \frac{1 - z_3^{T'/T}}{1 - z_3} z_3^{1-T'/T} (z_3^\mu - 1)^2 = \alpha_3,$$

et puis, en faisant les combinaisons des équations du (3.3), de la forme $n_i e_{m,n_j} - n_j e_{m,n_i}$, ($j = i + 1$; $i = 1, 2, 3, 4$), en vue de l'élimination de C'_4 , par des retranchements successifs et en faisant encore les notations

$$(3.8) \quad A = 2e_{m,\mu} - e_{m,2\mu}, \quad B = 2e_{m,2\mu} - e_{m,\mu} - e_{m,3\mu}, \\ C = 2e_{m,3\mu} - e_{m,2\mu} - e_{m,4\mu}, \quad D = 2e_{m,4\mu} - e_{m,3\mu} - e_{m,5\mu}$$

et

$$(3.9) \quad w_2 = z_2^\mu, \quad w_3 = z_3^\mu,$$

on arrive au système final de quatre équations aux quatre inconnues α_2 , α_3 , w_2 et w_3 ,

$$(3.10) \quad \begin{cases} \alpha_2 + \alpha_3 = A, & w_2^2 \alpha_2 + w_3^2 \alpha_3 = C, \\ w_2 \alpha_2 + w_3 \alpha_3 = B, & w_2^3 \alpha_2 + w_3^3 \alpha_3 = D, \end{cases}$$

où A , B , C et D sont des nombres positifs résultant des données expérimentales.

Il est facile de voir qu'on ne peut pas avoir $w_2 = w_3$, car cette supposition impliquerait d'abord

$$\frac{A}{B} = \frac{B}{C} = \frac{C}{D} = \frac{1}{w_2} = \frac{1}{w_3}$$

et ensuite, en vertu des équations du système,

$$A = B = C = D \quad \text{et} \quad w_2 = w_3 = 1.$$

Mais, compte tenu de (3.9) et (3.6), il résulterait $\tau_2 = \tau_3 = \infty$, possibilité qui peut être mise en discussion du point de vue physique.

En laissant de côté ce cas, on supposera dans la suite $w_2 \neq w_3$. Il résulte alors du (3.10),

$$(3.11) \quad \alpha_2 = \frac{Aw_3 - B}{w_3 - w_2} = \frac{Cw_3 - D}{w_2^2(w_3 - w_2)}, \quad \alpha_3 = \frac{B - Aw_2}{w_3 - w_2} = \frac{D - Cw_2}{w_3^2(w_3 - w_2)}$$

et

$$(3.12) \quad Bw_2w_3 - C(w_2 + w_3) + D = 0, \quad w_2^2(Aw_3 - B) = Cw_3 - D.$$

Le dernier système conduit à l'équation de deuxième degré

$$(3.13) \quad (B^2 - AC)w^2 + (AD - BC)w + C^2 - BD = 0,$$

dont les racines — qui sont toujours réelles, distinctes et positives — représentent la solution du système (3.12). Donc, en notant par w_2^0 et w_3^0 les racines de l'équation (3.13), la solution du système (3.10) est la suivante :

$$(3.14) \quad w_2 = w_2^0, \quad w_3 = w_3^0, \quad \alpha_2 = \alpha_2^0 = \frac{Aw_3^0 - B}{w_3^0 - w_2^0}, \quad \alpha_3 = \alpha_3^0 = \frac{B - Aw_2^0}{w_3^0 - w_2^0}.$$

Alors, par l'intermédiaire de (3.9) on obtient de (3.6) les valeurs des temps de retardation τ_2 et τ_3 ,

$$(3.15) \quad \tau_2 = \frac{\lambda_2}{E_2} = -\frac{\mu T}{\ln w_2^0}, \quad \tau_3 = \frac{\lambda_3}{E_3} = -\frac{\mu T}{\ln w_3^0}$$

et de (3.7) on obtient les valeurs de C_2 et C_3 :

$$(3.16) \quad C_j = \frac{\alpha_j^0 [1 - (w_j^0)^{1/\mu}] (w_j^0)^{(T'/T-1)/\mu}}{[1 - (w_j^0)^{T'/(\mu T)}] (1 - w_j^0)^2} = \frac{K \alpha p_0}{E_j}, \quad (j = 2, 3).$$

Par conséquent, les valeurs des modules d'élasticité des éléments du modèle considéré ont l'expression

$$(3.17) \quad E_j^0 = K \alpha p_0 \frac{[1 - (w_j^0)^{T'/(\mu T)}] (1 - w_j^0)^2}{\alpha_j^0 [1 - (w_j^0)^{1/\mu}] (w_j^0)^{(T'/T-1)/\mu}}, \quad (j = 2, 3).$$

Du (3.15) on obtient alors

$$(3.18) \quad \lambda_j = -\frac{\mu T E_j^0}{\ln w_j^0}, \quad (j = 2, 3).$$

Maintenant, on peut calculer C'_4 en utilisant la première équation de (3.3) et de son expression résulte la valeur de λ_4 ,

$$(3.19) \quad \lambda_4 = Kap_0 \mu T' \left[e_{m,\mu} + \frac{A(w_2^0 + w_3^0) - (A + B)}{1 + w_2^0 w_3^0 - (w_2^0 + w_3^0)} \right]^{-1}.$$

En conclusion, les éléments du modèle de Burgers à deux cellules sont donnés par les formules (3.5), (3.13), (3.14), (3.17)... (3.19).

4. La résolution du système pour le modèle de Burgers à trois cellules

Le système (1.1), (1.2) prend pour ce cas la forme

$$(4.1) \quad e_{M,n} = C_1 + \sum_{k=2}^4 C_k \frac{1 - e^{-T'/\tau_k}}{1 - e^{-T/\tau_k}} (1 - e^{-nT/\tau_k}) + C'_5 T' n,$$

$$(4.2) \quad e_{m,n} = \sum_{k=2}^4 C_k \frac{1 - e^{-T'/\tau_k}}{1 - e^{-T/\tau_k}} (1 - e^{-nT/\tau_k}) e^{-(T-T')/\tau_k} + C'_5 T' n,$$

et on se propose de lui trouver une solution dans les mêmes conditions qu'ont été imposées dans le paragraphe précédent.

Vu que E_1 est donné par

$$(4.3) \quad E_1 = \frac{Kap_0}{C_1^0},$$

où C_1^0 est la valeur de C_1 qui est obtenue du (4.1) après l'évaluation des autres inconnues, il reste à déterminer les inconnues

$$(4.4) \quad C_k = \frac{Kap_0}{E_k}, \quad \tau_k = \frac{\lambda_k}{E_k}, \quad (k = 2, 3, 4),$$

et

$$(4.5) \quad C'_5 = \frac{Kap_0}{\lambda_5}.$$

Dans ce but on considère un ensemble $\{n_k\}$, ($k = 1, 2, \dots, 7$), d'indices de la forme

$$(4.6) \quad n_k = k\mu, \quad (k = 1, 2, \dots, 7),$$

et on fait les notations suivantes :

$$(4.7) \quad A_j = 2e_{m,j\mu} - e_{m,(j+1)\mu} - e_{m,(j-1)\mu}, \quad (j = 1, 2, \dots, 6; e_{m,0} = 0),$$

$$(4.8) \quad w_k = e^{-\mu T/\tau_k}, \quad (k = 2, 3, 4),$$

$$(4.9) \quad \alpha_k = C_k \frac{1 - w_k^{T'/(\mu T)}}{1 - w_k^{1/\mu}} w_k^{(1-T'/T)/\mu} (1 - w_k)^2, \quad (k = 2, 3, 4).$$

Alors, les équations (4.2) écrites pour les valeurs (4.6) de n sont équivalentes au système à six inconnues α_k et w_k ,

$$(4.10) \quad \sum_{k=2}^4 w_k^{j-1} \alpha_k = A_j, \quad (j = 1, 2, \dots, 6),$$

qui a été obtenu de (4.2) par des combinaisons linéaires convenablement choisies.

En considérant (4.10) comme un système linéaire en α_2, α_3 et α_4 et en supposant $w_2 \neq w_3 \neq w_4$, les conditions de compatibilité écrites dans la forme

$$(4.11) \quad \begin{vmatrix} 1 & 1 & 1 & A_1 \\ w_2 & w_3 & w_4 & A_2 \\ w_2^2 & w_3^2 & w_4^2 & A_3 \\ w_2^{2+j} & w_3^{2+j} & w_4^{2+j} & A_{3+j} \end{vmatrix} = 0, \quad (j = 1, 2, 3),$$

conduisent au système linéaire

$$(4.12) \quad \begin{cases} A_3 S_1 - A_2 S_2 + A_1 S_3 = A_4, \\ A_4 S_1 - A_3 S_2 + A_2 S_3 = A_5, \\ A_5 S_1 - A_4 S_2 + A_3 S_3 = A_6, \end{cases}$$

où par S_i on a noté

$$(4.13) \quad \begin{aligned} S_1 &= w_2 + w_3 + w_4, & S_2 &= w_2 w_3 + w_2 w_4 + w_3 w_4, \\ S_3 &= w_2 w_3 w_4. \end{aligned}$$

En supposant le déterminant de ce système différent de zéro et en notant par (S_1^0, S_2^0, S_3^0) la solution correspondante, il s'ensuit que w_2, w_3, w_4 doivent être les racines de l'équation

$$(4.14) \quad w^3 - S_1^0 w^2 + S_2^0 w - S_3^0 = 0.$$

En notant par w_2^0, w_3^0 et w_4^0 , ($w_2^0 < w_3^0 < w_4^0$), les racines de l'équation (4.14), qu'on suppose réelles, distinctes et positives par des raisons suggérées par les données expérimentales, les trois premières équations de (4.10) admettent une solution unique comme fonction de ces valeurs, qu'on note par $(\alpha_2^0, \alpha_3^0, \alpha_4^0)$.

On obtient alors de (4.8) et (4.9) les formules

$$(4.15) \quad C_j = \frac{K\alpha_0}{E_j} = \frac{\alpha_j^0 [1 - (w_j^0)^{1/\mu}] (w_j^0)^{(T'/T-1)/\mu}}{[1 - (w_j^0)^{T'/(\mu T)}] (1 - w_j^0)^2}, \quad (j = 2, 3, 4),$$

et

$$(4.16) \quad \tau_j = \frac{\lambda_j}{E_j} = - \frac{\mu T}{\ln w_j^0}, \quad (j = 2, 3, 4),$$

qui livrent les valeurs cherchées pour E_j et λ_j , ($j = 2, 3, 4$). C_5 sera déterminé par l'une des équations initiales.

5. Observations finales

1) Pour qu'un modèle à une, deux ou trois cellules et une suite de données expérimentales soient compatible du point de vue mathématique, il est nécessaire et suffisant que les valeurs de w , données par (2.15), (3.13) et (4.14) respectivement, appartiennent à l'intervalle $]0, 1[$.

2) Les formules obtenues peuvent être programmées facilement au ordinateurs.

3) Pour trouver des valeurs des inconnues approchées de leurs valeurs exactes (du point de vue physique), il faut observer leur variation en fonction de μ et puis de décider les valeurs finales en appliquant — par exemple — la méthode des moindres carrés.

4) Vu qu'on peut étendre la méthode déjà exposée au cas des modèles à un nombre quelconque de cellules, on reviendra sur ce problème après l'obtention des résultats satisfaisants du point de vue expérimental.

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O REZOLVARE EXACTĂ A ECUAȚIILOR LUI VERSTRAETEN ȘI ROMAIN

(Rezumat)

Se dau o serie de formule exacte rezultând din rezolvarea ecuațiilor (1.1) și (1.2) ale lui Verstraeten și Romain, ce pot fi folosite pentru stabilirea unor caracteristici ale solurilor utilizate în construcția drumurilor. Formulele stabilite pot fi programate cu ușurință la calculatoare. De remarcat utilitatea și simplitatea formulelor obținute precum și metoda de eliminare concepută în scopul obținerii lor.

BOUNDARY LAYER EQUATIONS FOR MICROPOLAR FLUIDS

BY

R. S. NANDA

1. Introduction

Eringen [1]...[4] has recently introduced the theory of microfluids taking into account certain microscopic effects arising from the local structure and intrinsic motion of fluid elements, but nevertheless, retaining the continuum approach. Micropolar fluids [3], [4] are a special class of such fluids in which the local fluid elements in addition to the usual rigid motion, are allowed to undergo rotations in an average sense. It is expected that the study of such fluids may be useful in understanding the „physics“ of certain liquid crystals, fluids carrying additives and polymers. Many workers [5]...[9] have obtained simple exact solutions for a class of one-dimensional problems in which there is only one non-zero velocity component.

In the present paper we extend the concept of boundary layer to micropolar fluids to encompass a larger class of problems. The boundary layer equations for a two dimensional flow field are derived. Eringen [4] has shown that in the equations of motion of micropolar fluids, five new parameters are introduced in addition to the modified Reynolds number. In the case of classical viscous fluids there is only one parameter viz. the Reynolds number which controls the boundary layer.

In the present theory, we have additional parameters which are in the nature of degrees of freedom. By suitable choice of the relative orders of these parameters, the boundary layer equations can be further simplified. One such possibility is indicated in the present paper and it is shown how the known results from magnetohydrodynamics can be applied to micropolar fluids.

The concept of boundary layer has been successfully applied in the past to various classes of non-Newtonian and visco-elastic fluids [10]...[13].

2. Basic Equations

The constitutive equation for incompressible micropolar fluids as derived by Eringen, are

$$(1) \quad t_{rl} = -p\delta_{rl} + (2\mu + k)d_{rl} + k\varepsilon_{rln}(\omega_n - \sigma_n),$$

$$(2) \quad m_{rl} = \alpha\sigma_{nn}\delta_{rl} + \beta\sigma_{rl} + \gamma\sigma_{lr},$$

where

$$2d_{rl} = (v_{l,r} + v_{r,l}), \quad 2\omega_n = \varepsilon_{nlr}v_{r,l},$$

t_{rl} and m_{rl} are the stress tensor and the couple stress tensor, v_n is the velocity vector, σ_n is the microrotation vector and $\mu, k, \alpha, \beta, \gamma$ are constants characterising the fluid. ε_{nlr} is the alternating tensor and δ_{rl} is the Kronecker delta.

The above equations are to be used in conjunction with the equation of continuity

$$(3) \quad v_{i,i} = 0$$

and the conservation laws

$$(4) \quad t_{rl,r} + \rho(f_l - \dot{v}_l),$$

$$(5) \quad m_{rn,r} + \varepsilon_{nlr}t_{lr} + \rho(F_n - j\dot{\sigma}_n),$$

where f_l and F_n are the external body force and body couple respectively, both taken per unit mass, σ_k is the microrotation vector, ρ is the density, v_l and $\dot{\sigma}_n$ are the convected time derivatives.

The statement of the problem may be completed by specifying the boundary conditions. If v_B and σ_B are the prescribed values of the velocity and the microrotation, then the no slip conditions require $v = v_B$ and $\sigma = \sigma_B$ on a rigid boundary.

3. Boundary Layer Equations

Consider the flow of a micropolar fluid past a semi-infinite flat plate. Take the origin at the leading edge, the x -axis along the plate and the y -axis normal to it. We take

$$(6) \quad v = [u(x, y, t), \quad v(x, y, t), \quad 0], \quad \sigma = [0, 0, \Phi(x, y, t)].$$

Then the stress tensor and the couple stress tensor become

$$(7) \quad t_{xx} = -p + (2\mu + k) \frac{\partial u}{\partial x},$$

$$(8) \quad t_{xy} = \frac{1}{2}(2\mu + k)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 2\Phi\right),$$

$$(9) \quad t_{yx} = \frac{1}{2}(2\mu + k)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{1}{2}k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 2\Phi\right),$$

$$(10) \quad t_{yy} = -p + (2\mu + k)\frac{\partial v}{\partial y},$$

$$(11) \quad t_{zz} = -p, \quad t_{zx} = t_{xz} = t_{yz} = t_{zy} = 0,$$

$$(12) \quad m_{xz} = \gamma \frac{\partial \Phi}{\partial x}, \quad m_{zx} = \beta \frac{\partial \Phi}{\partial x},$$

$$(13) \quad m_{yz} = \gamma \frac{\partial \Phi}{\partial y}, \quad m_{zy} = \beta \frac{\partial \Phi}{\partial y},$$

$$(14) \quad m_{xx} = m_{yy} = m_{zz} = m_{xy} = m_{yx} = 0.$$

The stress equations of motion and the equation of continuity give

$$(15) \quad \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} t_{xx} + \frac{\partial}{\partial y} t_{yx},$$

$$(16) \quad \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} t_{xy} + \frac{\partial}{\partial y} t_{yy},$$

$$(17) \quad \rho j \left(\frac{\partial \Phi}{\partial t} + u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} \right) = \frac{\partial}{\partial x} m_{xz} + \frac{\partial}{\partial y} m_{yz} + t_{xy} - t_{yx},$$

and

$$(18) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

These equations can be simplified by introducing the concept of boundary layer. The basic simplifying assumptions of boundary layer theory are that the fluid velocity changes from zero at the surface of the plate to the free stream value U_∞ in a small distance $\delta(x, t)$, and that the gradient of any physical quantity along the normal is much greater than the gradient along the surface. The following assumptions about the orders of magnitude are made:

$$(19) \quad \begin{aligned} u &\sim O(1), & v &\sim O(\delta), \\ \frac{\partial}{\partial x} &\sim O(1), & \frac{\partial}{\partial y} &\sim O\left(\frac{1}{\delta}\right). \end{aligned}$$

These orders of magnitude are consistent with the equation of continuity, while Eqs. (15) and (16) reduce to

$$(20) \quad \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} t_{xx} + \frac{\partial}{\partial y} t_{yx},$$

$$(21) \quad 0 = - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} t_{xy} + \frac{\partial}{\partial y} t_{yy}.$$

Eliminating p by integrating Eqs. (21) from $y = 0$ to y and then differentiating with respect to x we have

$$(22) \quad \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \left(\frac{\partial}{\partial x} t_{yy} \right)_{y=0} + \frac{\partial}{\partial x} (t_{xx} - t_{yy}) + \frac{\partial}{\partial y} t_{yx}.$$

The term $\frac{\partial^2}{\partial x^2} \int_0^y t_{xy} dy$, which is of order $(t_{xy})_{y=0} \delta$ is neglected in comparison with the remaining terms. Taking the limit as $y \rightarrow \infty$, we have

$$(23) \quad \rho \left(\frac{\partial U_\infty}{\partial t} + U_\infty \frac{\partial}{\partial x} U_\infty \right) = \left(\frac{\partial}{\partial x} t_{yy} \right)_{y=0} + \left[\frac{\partial}{\partial x} (t_{xx} - t_{yy}) \right]_{y \rightarrow \infty}.$$

Subtracting (23) from (22) we have

$$(24) \quad \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho \left(\frac{\partial}{\partial t} U_\infty + U_\infty \frac{\partial}{\partial x} U_\infty \right) + \frac{\partial}{\partial x} (t_{xx} - t_{yy}) + \frac{\partial}{\partial y} t_{yx} - \left[\frac{\partial}{\partial x} (t_{xx} - t_{yy}) \right]_{y \rightarrow \infty}.$$

Retaining only terms of highest order in equations (7) to (13), substituting in equations (17) and (24) and then applying an order of magnitude analysis to eliminate the minor terms regardless of their origin, we have

$$(25) \quad \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho \left(\frac{\partial U_\infty}{\partial t} + U_\infty \frac{\partial U_\infty}{\partial x} \right) + (\mu + k)^2 \frac{\partial u}{\partial y^2} + k \frac{\partial \Phi}{\partial y},$$

$$(26) \quad \rho j \left(\frac{\partial \Phi}{\partial t} + u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} \right) = \gamma \frac{\partial^2 \Phi}{\partial y^2} - k \frac{\partial u}{\partial y} - 2k \Phi.$$

These are the boundary layer equations for micropolar fluids. It may be remarked here that k plays the role of a coupling constant. If k vanishes, then Eq. (25) reduces to the classical boundary layer equation.

4. Similarity Parameters

Introducing the dimensionless variables

$$\bar{u} = \frac{u}{U_0}, \quad \bar{v} = \frac{v}{U_0}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L},$$

$$\bar{\Phi} = \frac{\varphi}{\varphi_0}, \quad \bar{p} = \frac{p}{\rho U_0^2}, \quad \bar{t} = \frac{U_0 t}{L}, \quad \bar{U}_\infty = \frac{U_\infty}{U_0},$$

Eqs. (25) and (26) can be put in the form (dropping the bars)

$$(27) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\partial U_\infty}{\partial t} - U_\infty \frac{\partial U_\infty}{\partial x} = \frac{1}{\mathcal{R}} \frac{\partial^2 u}{\partial y^2} + \mathcal{T} \frac{\partial \Phi}{\partial y},$$

$$(28) \quad \frac{\partial \Phi}{\partial t} + u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} + 2E\Phi = \frac{1}{\mathcal{N}} \frac{\partial^2 \Phi}{\partial y^2} - \frac{\mathcal{M}}{\mathcal{N}} \frac{\partial u}{\partial y},$$

where

$$\mathcal{R} = \frac{U_0 L \rho}{\mu + k}, \quad \mathcal{T} = \frac{k \Phi_0}{\rho U_0^2}, \quad \mathcal{M} = \frac{k L U_0}{\gamma \Phi_0}, \quad \mathcal{N} = \frac{e j L U_0}{\gamma}, \quad \mathcal{E} = \frac{L k}{e j U_0}.$$

$\mathcal{R}$ is the modified Reynolds number while $\mathcal{T}$, $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{E}$ are new parameters controlling the boundary layer. If $\mathcal{N} \ll 1$, while $\mathcal{M}$ and $\mathcal{E}$ are of unit order the micro-inertia terms in Eq. (28) may be neglected. Then integrating once with respect to y we have

$$(29) \quad \frac{\partial \Phi}{\partial y} - \mathcal{M}(u - U_\infty) = 0.$$

Substituting this value in Eq. (27) we have

$$(30) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial t} U_\infty + U_\infty \frac{\partial U_\infty}{\partial x} + \frac{1}{\mathcal{R}} \frac{\partial^2 u}{\partial y^2} + \mathcal{T} \mathcal{M}(u - U_\infty).$$

An analogy can be found between this problem and the problem of magnetic boundary layer under the action of a transverse magnetic field; if the induced magnetic field is neglected, the magnetohydrodynamic boundary layer equation as derived by Rossow [14] is

$$(31) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U_\infty}{\partial t} + U_\infty \frac{\partial U_\infty}{\partial x} + \frac{1}{\mathcal{R}} \frac{\partial^2 u}{\partial y^2} - \mathcal{S} \mathcal{R}_m (u - U_\infty),$$

where $\mathcal{S}$ is the magnetic pressure number and $\mathcal{R}_m$ is the magnetic Reynolds' number.

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ECUAȚIILE STRATULUI LIMITĂ PENTRU FLUIDE MICROPOLARE

(Rezumat)

Se deduc ecuațiile de mișcare pentru curgerea bidimensională a unui fluid micropolar incompresibil și se extinde la aceste ecuații conceptul de strat limită. În general, ecuațiile stratului limită se prezintă ca un sistem cuplat de trei ecuații cu derivate parțiale în viteză și cîmpurile de microrotație. Ecuațiile devin independente atunci cînd se neglijează termenii corespunzători microinerției.

Se poate stabili o analogie între problema tratată și problema stratului limită magnetic sub acțiunea unui cîmp magnetic transversal.

ON ALMANSI PROBLEM FOR ORTHOTROPIC BEAMS

BY

C. I. BORȘ and GAB ELA S. NEAGU

1. Let us consider an orthotropic beam limited by two planes $x_3 = 0$, $x_3 = h$, ($h > 0$) and by a cylindrical surface $\mathcal{F}$.

The domain occupied by the beam will be denoted by $\mathcal{O}$, the domain of a cross-section of the beam and its area will be denoted by S and the boundary of S by Γ .

We take the x_3 -axis to be central-line of the beam and the axes x_1 and x_2 to be principal axes of inertia of the end $x_3 = 0$. In this case we have

$$(1) \quad \iint_S x_1 d\sigma = 0, \quad \iint_S x_2 d\sigma = 0, \quad \iint_S x_1 x_2 d\sigma = 0.$$

The relations between the components of strain γ_{ij} and the components of stress σ_{ij} are taken in the form

$$(2) \quad \begin{cases} E\gamma_{11} = \nu_{11}\sigma_{11} + \nu_{12}\sigma_{22} - \nu_1\sigma_{33}, \\ E\gamma_{22} = \nu_{12}\sigma_{11} + \nu_{22}\sigma_{22} - \nu_2\sigma_{33}, \\ E\gamma_{33} = -\nu_1\sigma_{11} - \nu_2\sigma_{22} + \sigma_{33}, \\ D\gamma_{12} = \sigma_{12}, \quad L\gamma_{23} = \sigma_{23}, \quad M\gamma_{31} = \sigma_{31}, \end{cases}$$

where ν_{ij} , ν_i , E , D , L , M are constants which characterize the elastic qualities of the material of the beam.

The components of strain γ_{ij} must satisfy the compatibility conditions of Saint - Venant given by

$$(3) \quad \begin{cases} \gamma_{11,22} + \gamma_{22,11} = \gamma_{12,12}, \dots \\ (-\gamma_{23,1} + \gamma_{31,2} + \gamma_{12,3}),_1 = 2\gamma_{11,23} \dots \end{cases}$$

We suppose that there are no body forces so that the equilibrium equations can be written in the form

$$(4) \quad \sigma_{ij,j} = 0 \quad \text{in } (\mathcal{O}^1).$$

We suppose that the tractions applied on the lateral surface are given by

$$(5) \quad \sigma_{ij} n_j = \sum_{k=0}^n \tau_{(i)}^k(x_1, x_2) x_3^k \quad \text{on } \mathcal{F},$$

where n_i are the directions cosines of the exterior normal to the surface $\mathcal{F}$ and $\tau_{(i)}^k$ are given functions of x_1 and x_2 . At the ends there are applied tractions in order to equilibrate the loads (5).

Here, we shall find a solution which satisfies the equations (4) and (5). After that it remains to satisfy the end conditions but this is the Saint - Venant problem and we know how to solve it [2].

The problem defined by (4) and (5) is known as Almansi's problem. Some methods are known to solve this problem for the isotropic case [1], [4] and for the orthotropic case [5].

Generally, in the known solutions a step by step method is given which allows to solve the problem $\sigma_{ij} \eta_j = \tau_i(x_1, x_2) x_3^{l+1}$ when the solution of problem $\sigma_{ij} \eta_j = \tau_i(x_1, x_2) x_3^k$ is known.

Here, a solution of the complete problem is given, such that its determination is equivalent to the calculus of the first step in the other known solutions.

2. We shall try to solve the Almansi problem supposing that the s stress components σ_{ij} are given by

$$(6) \quad \left\{ \begin{aligned} \sigma_{11} &= \sum_{k=0}^n \sigma_{11}^{(k)} x_3^k + \sum_{k=0}^{n-1} k \psi_{k,22} x_3^{k-1}, \\ \sigma_{22} &= \sum_{k=0}^n \sigma_{22}^{(k)} x_3^k - \sum_{k=0}^{n-1} k \psi_{k,11} x_3^{k-1}, \\ \sigma_{33} &= EA x_3 + \sum_{k=0}^n \left\{ \frac{E}{(k+1)(k+2)} \left[A_k + \frac{a_k}{\nu_2} x_1 + \frac{b_k}{\nu_1} x_2 \right] x_3^2 + \Omega_k \right\} x_3^k, \\ \sigma_{23} &= L \left[\sum_{k=0}^n \frac{1}{k+1} (\omega_k + \theta_k^{(2)}) x_3^{k+1} + \omega \right]_{,2} + \sum_{k=0}^{n-1} \psi_{k,112} x_3^k, \\ \sigma_{31} &= M \left[\sum_{k=0}^n \frac{1}{k+1} (\omega_k + \theta_k^{(1)}) x_3^{k+1} + \omega \right]_{,1} - \sum_{k=0}^{n-1} \psi_{k,122} x_3^k, \\ \sigma_{12} &= - \sum_{k=0}^n \Phi_{k,12} x_3^k, \end{aligned} \right.$$

where ω , ψ_k , ω_k , Φ_k are unknown functions of x_1 and x_2 ,

¹⁾ We use the summation convention over repeated indices. The index j after comma indicates partial differentiation with respect to x_j .

$$\begin{aligned}
\sigma_{11}^{(k)} &= \Phi_{k,22} - M(\omega_k + \theta_k^{(1)}), & \sigma_{22}^{(k)} &= \Phi_{k,11} - L(\omega_k + \theta_k^{(2)}), \\
\theta_k^{(1)} &= a_k x_1 x_2^2 - c_k x_1 x_2, & \theta_k^{(2)} &= b_k x_1^2 x_2 + c_k x_1 x_2, \\
\Omega_k &= E \left[\omega_k + \frac{1}{2} (\theta_k^{(1)} + \theta_k^{(2)}) + \frac{1}{6} \left(\frac{\nu_1}{\nu_2} a_k x_1^3 + \frac{\nu_2}{\nu_1} b_k x_2^3 \right) \right] + \\
&\quad + \nu_1 \sigma_{11}^{(k)} + \nu_2 \sigma_{22}^{(k)} + \frac{EA_k}{2} (\nu_1 x_1^2 + \nu_2 x_2^2) + \\
&\quad + \frac{E(k+1)}{2} \left[\left(\frac{2\nu_1}{E} - \frac{1}{M} \right) \psi_{k+1,22} - \left(\frac{2\nu_2}{E} - \frac{1}{L} \right) \psi_{k+1,11} + \frac{k+2}{D} \Phi_{k+2} \right], \\
(k &= 0, 1, 2, \dots, n); & \Phi_{n+1} &= \Phi_{n+2} = \psi_n = \psi_{n+1} = \Omega_{n+1} = 0
\end{aligned}$$

and A_k, a_k, b_k, c_k, A are constants which must be determined.

The stresses (6) will satisfy the equilibrium equations if the functions ω and ω_k verify the equations

$$(7) \quad \Delta \omega + EA + \Omega_1 = 0 \quad \text{in } S,$$

$$(8) \quad \Delta \omega_l + E \left(\frac{a_l}{\nu_2} x_1 + \frac{b_l}{\nu_1} x_2 + A_l \right) + (l+1)(l+2)\Omega_{l+2} = 0 \quad \text{in } S, \\
(l = 0, 1, \dots, n),$$

where the operator Δ is given by

$$(9) \quad \Delta = M \frac{\partial^2}{\partial x_1^2} + L \frac{\partial^2}{\partial x_2^2}.$$

Now, let us examine the compatibility conditions (3). The sixth condition is identically satisfied. The first condition will be satisfied if the function Φ_k verifies

$$\begin{aligned}
&\beta_{22} \frac{\partial^4 \Phi_k}{\partial x_1^4} + \beta_{11} \frac{\partial^4 \Phi_k}{\partial x_2^4} + (2\beta_{12} + \beta_{33}) \frac{\partial^4 \Phi_k}{\partial x_1^2 \partial x_2^2} = \frac{\partial^2 \omega_k}{\partial x_1^2} [M\beta_{12} + L\beta_{22} + \nu_2] + \\
&+ \frac{\partial^2 \omega_k}{\partial x_2^2} [M\beta_{11} + L\beta_{12} + \nu_1] + 2a_k x_1 [M\beta_{11} + \nu_1] + 2b_k x_2 [L\beta_{22} + \nu_2] + \\
(10) \quad &+ 2\nu_1 \nu_2 A_k + (k+1) \left[\left(\beta_{22} + \frac{\nu_2}{2L} \right) \frac{\partial^4 \psi_{k+1}}{\partial x_1^4} - \left(\beta_{11} + \frac{\nu_1}{2M} \right) \frac{\partial^4 \psi_{k+1}}{\partial x_2^4} + \right. \\
&\quad \left. + \left(\frac{\nu_1}{2L} - \frac{\nu_2}{2M} \right) \frac{\partial^4 \psi_{k+1}}{\partial x_1^2 \partial x_2^2} + \frac{k+2}{2\Delta} \left(\frac{\partial^2 \Phi_{k+2}}{\partial x_1^2} \nu_2 + \nu_1 \frac{\partial^2 \Phi_{k+2}}{\partial x_2^2} \right) \right],
\end{aligned}$$

where

$$\beta_{ij} = \frac{\nu_{ij} - \nu_i \nu_j}{E}, \quad (i, j = 1, 2); \quad \beta_{33} = \frac{1}{\Delta}.$$

The fourth and fifth compatibility conditions lead to

$$(11) \quad \frac{\partial}{\partial x_1} \left[\frac{\partial^2}{\partial x_1 \partial x_2} (\Delta \psi_k) \right] = \Theta_{k+1,2}^{(1)}, \quad \frac{\partial}{\partial x_2} \left[\frac{\partial^2}{\partial x_1 \partial x_2} (\Delta \psi_k) \right] = \Theta_{k+1,1}^{(2)},$$

where

$$\begin{aligned} \Theta_{k+1}^{(1)} &= -\frac{2(k+1)LM}{E} \left[\nu_{11} \sigma_{11}^{k+1} + \gamma_{12} \sigma_{22}^{k+1} - \nu_1 \Omega_{k+1} + \frac{E}{2\Delta} \Phi_{k+1,11} + \right. \\ &\quad \left. + (k+2)(\nu_{11} \psi_{k+2,22} - \nu_{12} \psi_{k+2,11}) \right], \\ \Theta_{k+1}^{(2)} &= \frac{2(k+1)LM}{E} \left[\nu_{12} \sigma_{11}^{k+1} + \nu_{22} \sigma_{22}^{k+1} - \nu_2 \Omega_{k+1} + \frac{E}{2D} \Phi_{k+1,22} + \right. \\ &\quad \left. + (k+2)(\nu_{12} \psi_{k+2,22} - \nu_{22} \psi_{k+2,11}) \right]. \end{aligned}$$

From (11) we can get

$$(12) \quad \frac{\partial^2}{\partial x_1 \partial x_2} (\Delta \psi_k) = \Theta_{k+1},$$

where

$$\Theta_{k+1} = \int \Theta_{k+1,2}^{(1)} dx_1 = \int \Theta_{k+1,1}^{(2)} dx_2,$$

because the condition of integrability are satisfied in account of (10). From the second and third conditions we get

$$(13) \quad \frac{\partial^2}{\partial x_1^2} (\Delta \psi_k) = \Theta_{k+1}^{(1)}, \quad \frac{\partial^2}{\partial x_2^2} (\Delta \psi_k) = \Theta_{k+1}^{(2)}.$$

Taking into account (12) and (13) it is easy to prove that all required integrability conditions are satisfied and consequently we have

$$(14) \quad \Delta \psi_k = \iint_S \Theta_{k+1} dx_1 dx_2 \quad \text{in } S.$$

Thus, anywhere in this paper ψ_k is a solution of the equation (14) of the class C^4 . Now, let us consider the boundary conditions (5). The third boundary condition will be satisfied if the functions ω and ω_k satisfy the following conditions

$$(15) \quad D\omega = \tau_{(3)}^0 + \psi_{0,122} n_1 - \psi_{0,112} n_2, \quad \text{on } \Gamma,$$

$$(16) \quad \begin{aligned} D\omega_k &= -(M\Theta_{k,1}^{(1)} n_1 + L\Theta_{k,2}^{(2)} n_2) + \\ &+ (k+1)(\tau_{(3)}^{k+1} + \psi_{k+1,122} n_1 - \psi_{k+1,112} n_2) \quad \text{on } \Gamma, \end{aligned}$$

where

$$D = Mn_1 \frac{\partial}{\partial x_1} + Ln_2 \frac{\partial}{\partial x_2}.$$

The necessary and sufficient conditions to solve the boundary value problems (7), (15) and (8), (16) require that

$$A_k = -\frac{k+1}{ES} \left[\int_S \tau_{(3)}^{k+1} ds + (k+2) \iint_S \Omega_{k+2} d\sigma \right], \quad (k=0, 1, \dots, n),$$

$$A = -\frac{1}{ES} \left[\int_{\Gamma} \tau_{(3)}^0 ds + \iint_S \Omega_1 d\sigma \right].$$

The two equations (5) will be satisfied if the function Φ_k is chosen so that

$$\begin{aligned} \Phi_{k,22} n_1 - \Phi_{k,12} n_2 &= \tau_{(1)}^k + n_1 [M(\omega_k + \Theta_k^{(1)}) - (k+1) \psi_{k+1,22}] - \\ - \Phi_{k,12} n_1 + \Phi_{k,11} n_2 &= \tau_{(2)}^k + n_2 [L(\omega_k + \Theta_k^{(2)}) + (k+1) \psi_{k+1,11}] \quad \text{on } \Gamma. \end{aligned}$$

It is obvious that we can obtain the function Φ_k in a similar way to that corresponding to the determination of Airy's function in the plane problem of orthotropic bodies [2]. Using the same method as in [3] from the conditions of existence of function Φ_k we can determine the constants a_k, b_k, c_k before to know the function ω_k .

At the last we obtain

$$\begin{aligned} a_k &= -\frac{\nu_2}{EI_1} \left[\int_{\Gamma} \tau_{(1)}^k ds + (k+1) \int_{\Gamma} x_1 \tau_{(3)}^{k+1} ds + (k+1)(k+2) \iint_S x_1 \Omega_{k+2} d\sigma \right], \\ b_k &= -\frac{\nu_1}{EI_2} \left[\int_{\Gamma} \tau_{(2)}^k ds + (k+1) \int_{\Gamma} x_2 \tau_{(3)}^{k+1} ds + (k+1)(k+2) \iint_S x_2 \Omega_{k+2} d\sigma \right], \\ c_k &= \frac{1}{D_t} \left\{ \int_{\Gamma} (x_2 \tau_{(1)}^k - x_1 \tau_{(2)}^k) ds + (k+1) \int_{\Gamma} \varphi \tau_{(3)}^{k+1} ds + \iint_S \left\{ \varphi \left[E \left(\frac{a_k}{\nu_2} x_1 + \right. \right. \right. \right. \\ &+ \left. \frac{b_k}{\nu_1} x_2 + A_k \right) + (k+1)(k+2) \Omega_{k+2} \Big] + \left(x_2 - \frac{\partial \varphi}{\partial x_1} \right) \left(Ma_k x_2^2 - (k+1) \frac{\partial^3 \psi_{k+1}}{\partial x_2^2 \partial x_1} \right) - \\ &\left. - \left(x_1 + \frac{\partial \varphi}{\partial x_2} \right) \left(Lb_k x_1^2 + (k+1) \frac{\partial^3 \psi_{k+1}}{\partial x_1^2 \partial x_2} \right) \right\} d\sigma \Big\}, \end{aligned}$$

where φ is the function of torsion defined by

$$\Delta\varphi = 0 \quad \text{in } S,$$

$$D\varphi = Mx_2 n_1 - Lx_1 n_2 \quad \text{on } \Gamma,$$

$$I_i = \iint_S x_j^2 d\sigma, \quad (i, j = 1, 2; i \neq j),$$

$$D_i = \iint_S (Lx_1^2 + Mx_2^2 + Lx_1 \varphi_{,2} - Mx_2 \varphi_{,1}) d\sigma.$$

It is known [2] that $D_i > 0$ so that all functions and constants are determined.

It is easy to see that we must start in the following order: A_n , a_n , b_n , c_n , ω_n , Φ_n , A_{n-1} , a_{n-1} , b_{n-1} , c_{n-1} , ω_{n-1} , Φ_{n-1} , ψ_{n-1} , A , ω and so on.

In this way the problem of Almansi (4), (5) is solved.

The solution given here is a natural generalization of [3]. When $n = 0$ we obtain the results of [3]. By adequate particularizations we can obtain the isotropic case.

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ASUPRA PROBLEMEI LUI ALMANSI PENTRU BARE ORTOTROPE

(Rezumat)

Se dă o soluție pentru problema lui Almansi definită de (4) și (5). Soluția prezentată este cea mai simplă din cele cunoscute până acum.

Secția I

MATEMATICĂ

MECANICĂ TEORETICĂ, FIZICĂ

D. C. 621.319.1

L'ANISOTROPIE INDUITE AUX POUDRES DE MAGNÉTITE ¹⁾

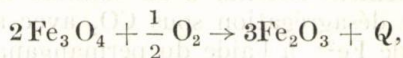
PAR

EMIL LUCA, C. NAUM, PAMFILIA PIETRARIU et IRINA JEMNA

On sait que certains matériaux ferromagnétiques réagissent au traitement thermique en champ magnétique (TTM) par l'apparition d'une anisotropie uniaxe induite au long du champ magnétique de traitement. La magnétite est un cas typique et nous allons analyser l'anisotropie induite dans le troisième domaine de relaxation en fonction de l'excès d'oxygène jusqu'à la transformation totale de la magnétite en $\gamma\text{-Fe}_2\text{O}_3$.

La relaxation de la magnétite a longtemps été une question pas assez élucidée, Braginski et autres [1] ont communiqué l'existence de trois maximums dans le spectre de désaccommodation de la magnétite. Nos résultats, ainsi que ceux des autres auteurs [14], concernant la variation de K_u^{III} avec la composition pour le système $\text{Fe}_x\text{Fe}_{3-x}\text{O}_{4+\delta}$ infirment tant le modèle de Motzke [2], [3] que de celui Braginski [15].

La magnétite est un spinelle inverse [4]...[6], les sites A étant occupés par des ions trivalents tandis que les sites B sont occupés une moitié par les ions bivalents et l'autre moitié par des ions trivalents, sans aucun ordre en ce que concerne la répartition des deux sorts d'ion. Quoiqu'elle appartient au système cubique, la magnétite présente des courbes d'aimantation différentes suivant les directions cristallographiques. L'axe d'aimantation facile est au long de la direction quaternaire et l'anisotropie magnétocristalline intervient dans les divers processus. La réaction exothermique d'oxydation de la magnétite est la suivante :



¹⁾ Présenté lors de la Session scientifique jubilaire de la Faculté d'Électrotechnique de Jassy, le 21—23 décembre 1972.

l'oxydation étant en directe dépendance des conditions de préparation et des dimensions des grains de magnétite. L'oxydation est progressive et fait apparaître une série de phases intermédiaires entre la magnétite et le $\gamma\text{-Fe}_2\text{O}_3$, sous 350°C , parce que au-delà de cette température la magnétite se transforme en $\alpha\text{-Fe}_2\text{O}_3$ nonmagnétique.

Hagg [7], fondé sur les constatations concernant les paramètres du réseau, admet que l'oxydation se produit par l'apparition des lacunes de fer, d'après le schéma : $3\text{Fe}^{2+} \rightarrow 2\text{Fe}^{3+} + \text{lacune}$. Les phases intermédiaires entre la magnétite et le $\gamma\text{-Fe}_2\text{O}_3$ peuvent être considérées une solution solide de deux oxydes. On peut supposer le mécanisme suivant pour l'oxydation [8] : la molécule d'oxygène adsorbé à la surface se transforme, par capture d'électrons, en O^{2-} qui se place sur la surface du réseau. Les électrons proviennent de la transformation $\text{Fe}^{2+} \rightarrow \text{Fe}^{3+} + e^-$ et arrivent à la surface par diffusion. Les sites cationiques vides auprès de O^{2-} seront occupés par les ions de fer placés à l'intérieur.

En même temps que l'oxydation, la constante du réseau change sa valeur de $8,378 \text{ \AA}$ à $8,322 \text{ \AA}$, ce que produit la modification de la cellule élémentaire de la magnétite ; sur la structure de la magnétite apparaît une superstructure tétragonale et la distribution des ions de Fe^{3+} étant uniforme impose un changement continu de position entre les ions de fer, ce que conduit à un rangement partiel et à l'apparition d'une anisotropie induite après TTM. Il y arrive un changement de position des vacances par les sauts des ions de fer.

Nous avons obtenu la poudre de magnétite par deux méthodes de préparation :

- a) décomposition thermique de l'oxalat ferreux [9] ;
- b) précipitation alcaline des sels ferreux suivie de l'oxydation [10], [8].

A) La décomposition thermique de l'oxalat ferreux a été faite dans des tubes en pyrex fermés, prévoyés avec deux capillaires, un de $0,5 \text{ mm}$ diamètre, dont l'orifice était fermé à l'aide d'un fil en platine de $0,49 \text{ mm}$ diamètre et l'autre de 1 mm diamètre, au début fermé, mais ouvert 16 heures après le commencement de la réaction. Le fil en platine pouvait être introduit dans le premier capillaire sur une distance convenable pour régler l'accès de l'air dans le tube de réaction. La réaction se passait à température constante, $350 \pm 20^\circ\text{C}$, dans un four à réglage automatique.

B) Dans la préparation de la magnétite K a u f f m a n n - H a b e r ont été variés certains paramètres, en rapport avec le procédé antérieur, pour obtenir une meilleure pureté et des dimensions des grains de magnétite plus convenables au but envisagé.

La magnétite tellement obtenue a été soumise à l'analyse chimique ; on a dosé le Fe^{2+} par désagrégation sous CO_2 avec acid phosphorique et puis on a titré le ion de Fe^{2+} à l'aide du permanganat de potassium [11].

L'oxydation de la magnétite a été faite sous volume limité d'air en fermant divers quantités de poudre dans des ampoules en verre, soumises ultérieurement au réchauffement à 250°C durant quelques heures.

Les échantillons mesurés ont été disques de 11 mm diamètre et de 4–6 mm épaisseur, la poudre étant stabilisée à l'aide de la paraffine (point de fusion 72°C).

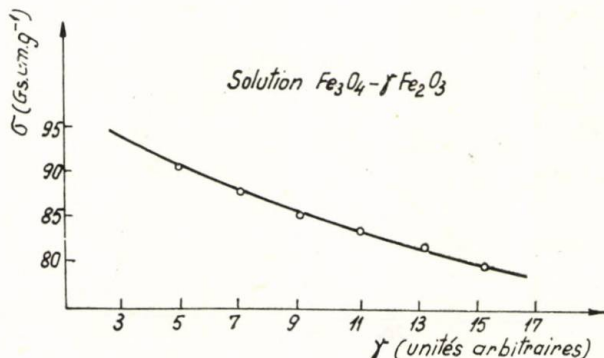


Fig. 1. — La variation de l'aimantation de saturation en fonction du degré d'oxydation.

Le traitement a été conduit dans un champ magnétique de 10 kOe, à la température ambiante de 23°C. Le measurement de l'anisotropie induite a été fait en régime isothermique, c'est à dire après TTM on tournait rapidement l'électro-aimant de 45° et on traceait à température constante la courbe du décroissement de la torsion en temps à l'aide d'un magnétomètre de torsion automatique [12].

On a mesuré aussi l'aimantation de saturation spécifique à l'aide d'un magnétomètre à échantillon vibrant et compensation automatique.

Dans la fig. 1 on peut voir la variation de l'aimantation pour les échantillons mesurés à 23°C, échantillons qui représentaient les différentes phases de la transformation de la magnétite en γ -Fe₂O₃. Les valeurs de δ décroissent quand le degré d'oxydation croît.

Fondé sur la structure proposée par Verwey et Boer [6], Néel explique la valeur de 98,23 Gs.cm.g⁻¹ correspondante à la magnétite et aussi la valeur de 83,5 pour γ -Fe₂O₃ déterminé par Weiss et Forrer [13].

Dans la fig. 2 est présentée la variation de l'anisotropie induite K_u^{III} dans le domaine III de relaxation. La courbe en pointillé représente la variation de K_u^{III} en fonction de l'excès d'oxygène pour la ferrite Mn_{0,1}Fe_{2,9}O₄ qui est une solution solide de magnétite et d'oxyde de manganèse dans laquelle on a substitué le Fe²⁺ avec Mn²⁺, d'après Gh. Maxim [14]. La courbe en trait plein représente la même variation de K_u^{III} en fonction de l'excès d'oxygène pour magnétite et ses diverses phases d'oxydation jusqu'à γ -Fe₂O₃, d'après nos résultats expérimentaux.

La croissance presque linéaire de K_u^{III} avec la concentration des vacances cationiques, dans le domaine des faibles oxydations, montre que cette anisotropie est donnée par l'arrangement des vacances uniques

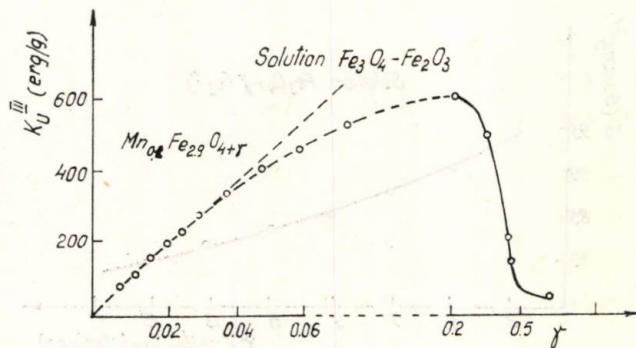


Fig. 2. — La variation de la constante K_u^{III} dans le domaine III de relaxation.

sur les quatre sortes de sites B . Le décroissement de K_u^{III} dans le domaine des oxydations fortes peut être expliqué par une diminution de l'énergie élémentaire d'anisotropie des vacances. γ pourrait intervenir l'arrangement à distance qui se produit dans le réseau octaédrique de $\gamma\text{-Fe}_2\text{O}_3$.

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ANIZOTROPIA INDUSĂ A PULBERII DE MAGNETITĂ

(Rezumat)

Se studiază variația anizotropiei induse în domeniul *III* de relaxare a magnetitei și a diverselor ei faze de oxidare pînă la $\gamma\text{-Fe}_2\text{O}_3$, în funcție de gradul de oxidare și magnetizarea de saturație specifică.

Pe baza rezultatelor experimentale se dă o explicație a mecanismului apariției anizotropiei uniaxiale induse și a variației acestei anizotropii în domeniul *III* de relaxare.

Rezultatele experimentale, mai ales cele legate de oxidări puternice, infirmă atit modelul lui *M o t z k e* cit și cel al lui *B r a g i n s k i*.

THE HISTORY OF THE UNITED STATES

The history of the United States is a story of growth and change. It begins with the first settlers who came to the continent in search of a new life. They found a land of opportunity, but also a land of challenge. The early years were marked by conflict and struggle, but the spirit of the American people was one of resilience and determination. Over time, the United States grew from a small colony into a great nation, one that has shaped the world in many ways. The story of the United States is a story of hope and dreams, of a people who have built a nation that stands for freedom and justice for all.

SOME MAGNETIC AND ELECTRIC CHARACTERISTICS OF Ni—Zn FERRITES WITH CURIE TEMPERATURE IN THE DOMAIN OF 0°C

BY

C. PASNICU, N. REZLESCU and EMIL LUCA

1. Introduction

The aim of the present paper is to investigate some electric and magnetic characteristics of Ni—Zn ferrites with the Curie point about 0°C: resistivity, saturation magnetization and the variation of the maximum induction with temperature. It is a continuation of paper [1] in which we investigated the dependence of the Curie temperature and of the initial permeability of the Ni—Zn ferrites on the composition and cooling method. In that paper [1] we have also shown that the ferrite with the Curie temperature in the domain of 0°C is the one having the $\text{Ni}_{0,23}\text{Zn}_{0,77}\text{Fe}_2\text{O}_4$ stoichiometric composition; in the present paper the researches have been carried out only on ferrites of a composition close to the one mentioned above.

2. Measuring Method

The samples, under the form of tori and pastilles, were prepared by the known ceramic technique starting from oxides: NiO, ZnO and Fe_2O_3 . The sinterization was made at a temperature of 1300°C for 4 hours. The samples were cooled slowly, from the sinterization temperature, in the furnace, at about 80°C/hour.

The resistivity was measured on samples under the form of pastilles, by using a Wheatstone bridge. A layer of Ag was deposited on the faces of the pastilles by thermic evaporation in vacuum. The resistivity values are given for the room temperature.

The saturation magnetization was measured at a temperature of about -180°C on samples under the form of spheres, by using the method of the vibrating sample described by Foner in 1959 [2]. The spheres

were obtained from the samples under the form of pastilles through Bond method [3].

The maximum induction was measured from the hysteresis curve, raised in a 50 Hz alternating current using an „Orion“ Ferotester. The sample was sunk in liquid azote and then raised gradually from the surface of the liquid azote; thus its temperature was increased continually. On the cycle the decrease of the maximum induction was watched.

3. Results and Discussions

3.1. Resistivity

We know that the conduction process lays at the basis of the so-called Verwey conduction mechanism” [4] which consists in the electronic exchange among the different valencies Fe ions from the octaedic places, $\text{Fe}^{2+} \rightleftharpoons \text{Fe}^{3+}$ [5]. So that greater is the quantity of Fe^{2+} ions in the ferrites, more intense is the conduction process and smaller the resistivity.

The dependence of the resistivity on the NiO and ZnO quantity for the whole system of stoichiometric Ni—Zn is shown in Fig. 1. We can see that as the ZnO quantity increases the resistivity decreases, at first slowly, then rather quickly till $x = 0,6$ from where the decrease is slow again. The resistivity decreases during this interval more than 10^3 times. This decrease of the resistivity might be explained through the evapora

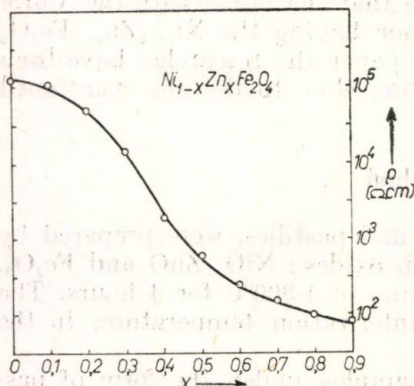


Fig. 1

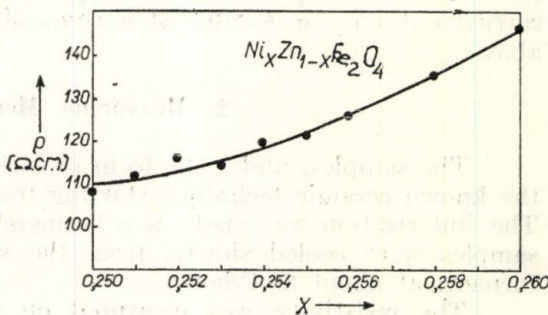


Fig. 2

tion phenomenon of Zn at the surface of the ferrite during the sinterization thermic treatment [6]. The evaporation of Zn leads to the formation of a corresponding quantity of Fe^{2+} ions in the ferrite (even for an initial stoichiometric composition). So that greater the ZnO quantity in the mixed

ferrite is, more intense the Zn evaporation is, and, consequently, the Fe^{2+} ions quantity will be greater as well; this will lead to the decrease of the resistivity.

Fig. 2 presents the dependence of the resistivity on small variations of the NiO and ZnO content about the chosen composition, when the Fe_2O_3 content remains constant. We notice that when the composition varies from $\text{Ni}_{0,25}\text{Zn}_{0,75}\text{Fe}_2\text{O}_4$ to $\text{Ni}_{0,26}\text{Zn}_{0,74}\text{Fe}_2\text{O}_4$ the resistivity increases very little, from 110 Ωcm to 145 Ωcm . This increase might be explained through the small reduction of the Fe^{2+} ions quantity as a result of the decrease of ZnO content in the ferrite.

Fig. 3 shows the same dependence when the ZnO content remains constant. In this case too, though the ZnO quantity remains constant, the resistivity increases quite much, from 32 Ωcm to 120 Ωcm , passing from $\text{Ni}_{0,155}\text{Zn}_{0,745}\text{Fe}_{2,095}\text{O}_4$ composition to $\text{Ni}_{0,255}\text{Zn}_{0,745}\text{Fe}_{2,015}\text{O}_4$ composition.

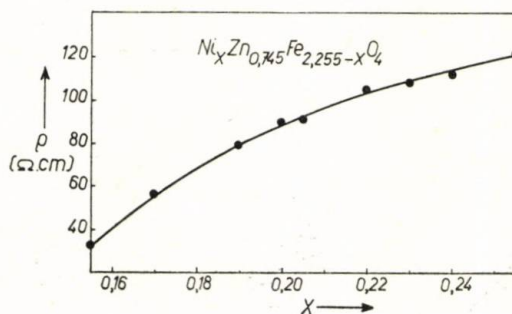


Fig. 3

This increase is also explained by the lessening of the number of Fe^{2+} ions (we must underline that the Fe excess above the initial stoichiometric composition is found again in the final composition under the form of Fe^{2+} so that in our discussions we can use either the expression excess of Fe, or Fe^{2+} ions quantity).

3.2. Saturation Magnetization

The Ni—Zn stoichiometric ferrites have the following distribution of ions :



(The ions placed on the octaedic places were put between parentheses).

We know [6] that the ferromagnetism is a „noncompensated” antiferromagnetism: the magnetization of the two sublattices octaedic and tetraedic of opposed signs, are not equal between themselves as in a common antiferromagnetic; difference of magnetization of sublattices pro-

duces increases (or decreases) of the lattice magnetization; the resultant magnetization increases.

In the case of Ni—Zn ferrites, the introduction of nonmagnetic Zn ions, which settle preferentially on the tetraedric places, and take the place of the Fe (Fe^{3+}) ions with the 5 magnetic moment on these places, will decrease the magnetization of the tetraedric sublattice and will increase the ferrite saturation magnetization (Fig. 4).

3.3. Variation of the Maximum Induction with Temperature

As we can see from Fig. 5 in which this variation is given, the maximum inductance decreases slowly alongside with the increase of the temperature to the Curie temperature where it shows a quick decrease. After passing over the Curie temperature the maximum induction does

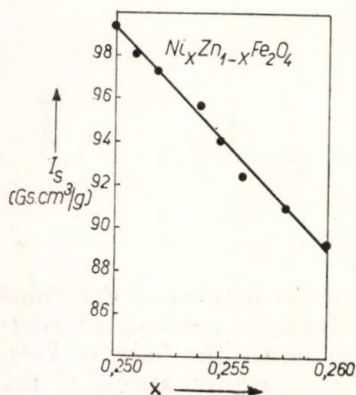


Fig. 4

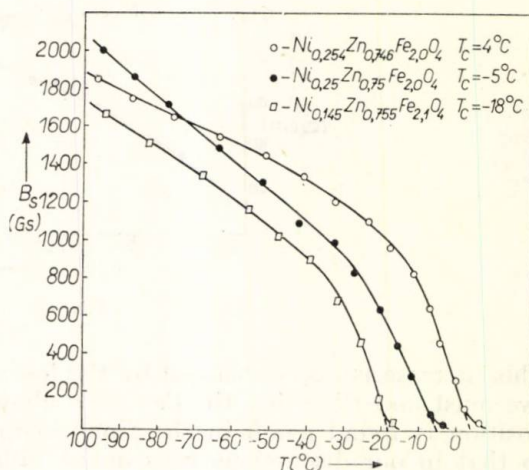


Fig. 5

not decrease to the 0 value as we should expect according to the theory. Our opinion is that this can be only explained by a „rest“ of spontaneous magnetization owing to the nonhomogeneous structure, passing from the exterior to the interior of the ferrite, a structure which it gets during the sinterization.

By the extrapolation towards zero of the portion of quick decrease of the curves given in Fig. 5, we can determine Curie temperature. The Curie temperature obtained in this way is, within the limits of the measurements errors, the same as the one obtained from the variation curves

of the initial permeability with temperature, that were given in paper [1]. We should not be surprised by this thing if we take into account that the measurements were made on some samples under the form of tori.

4. Conclusions

1. The element influencing to the greatest extent upon resistivity is the Fe^{2+} ions quantity, obtained in the final product either through the Fe_2O_3 excess above the initial stoichiometric composition, or through the evaporation process of Zn at the ferrite surface during the sinterization thermic treatment.

2. The saturation magnetization of the Ni—Zn ferrites depends on the nonmagnetic Zn^{2+} ions quantity from the tetraedric sublattice, because it is known that the magnetization of a ferrite is equal to the difference of the magnetization of the two sublattices, octaedric and tetraedric.

3. The maximum induction decreases continually as the temperature increases; but close to the Curie point this decrease is more rapid; thus we can determine the Curie temperature.

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UNELE CARACTERISTICI MAGNETICE ȘI ELECTRICE ALE FERITELOR DE Ni—Zn CU TEMPERATURA CURIE ÎN DOMENIUL LUI 0°C

(Rezumat)

Se studiază rezistivitatea și magnetizarea de saturație funcție de conținutul de NiO , ZnO , Fe_2O_3 și de asemenea variația inducției maxime cu temperatura pentru ferite cu punctul Curie în jur de 0°C, care în condiții stoechiometrice au aproximativ compoziția $\text{Ni}_{0.253}\text{Zn}_{0.47}\text{Fe}_2\text{O}_4$.

of the medical profession and the public. It is the duty of the medical profession to keep the public informed of the latest developments in medicine and surgery. The American Medical Association is the only organization that does this.

THE JOURNAL

The Journal of the American Medical Association is a weekly publication that contains the latest news and information in the medical field. It is a valuable resource for medical professionals and the general public alike. The Journal covers a wide range of topics, including medical research, clinical practice, and public health. It is published by the American Medical Association, which is the largest and most influential organization in the medical profession in the United States.

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THE JOURNAL OF THE AMERICAN MEDICAL ASSOCIATION
PUBLISHED WEEKLY
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The Journal of the American Medical Association is a weekly publication that contains the latest news and information in the medical field. It is a valuable resource for medical professionals and the general public alike.

SUR L'INFLUENCE DE LA NATURE DES ÉLECTRODES MÉTALLIQUES DANS LES PROCESSUS DE POLARISATION ET DE FORMATION DES ÉLECTRETS

PAR

EUGEN NEAGU, RODICA NEAGU et EMIL LUCA

1. Généralités

La méthode la plus employée pour obtenir des électrets est la polarisation d'un diélectrique chauffé jusqu'à une certaine température puis refroidi dans la présence d'un champ électrique d'une certaine valeur.

Cette méthode a été utilisée pour la première fois par E g u c h i [1] pour obtenir les premiers thermoélectrets en cire de Carnauba.

Pour expliquer le mécanisme de la formation des électrets ont été émises plusieurs théories [2]...[6], la plupart considérant les aspects phénoménologiques du décroissement avec le temps des charges superficielles des électrets, dont la valeur et la variation en temps dépendraient seulement de la nature de la substance, de la température d'échauffement et de l'intensité du champ électrique polarisant.

Parallèlement à ces théories, où le processus de décroissement de charges superficielles des électrets en temps était expliqué qualitativement, P e r l m a n [7] a formulé une théorie nonisotherme du décroissement des charges des électrets, fondée sur des résultats quantitatifs.

Recherches récentes [8] ont démontré que la nature des électrodes métalliques utilisés dans le processus de formation des électrets, a aussi, une importante influence en ce que concerne la nature, la valeur et le décroissement en temps des charges des électrets. Puisque dans les travaux cités l'influence de la nature des électrodes utilisés est considérée seulement dans le processus de formation des électrets sous l'action du champ électrique, nous nous proposons de déterminer séparément, dans la présence et dans l'absence du champ électrique, comment le processus de polarisation d'un diélectrique chauffé est influencé par la nature des électrodes métalliques utilisés, c'est à dire nous nous proposons de déterminer l'influence du contact métallo-diélectrique et puis comparer les résultats obtenus.

Dans le présent travail nous présentons seulement quelques résultats expérimentaux concernant l'utilisation des électrodes en Fe, Al, Cu et bronze.

2. Résultats expérimentaux

Pour obtenir les électrets nous avons employé la cire de Carnauba et le plexiglas, substances connues pour leurs propriétés d'électrets.

Les échantillons en cire de Carnauba, sous forme de disques de 1,6 mm épaisseur, ont été obtenus à l'aide d'une matrice de 25 mm diamètre utilisant la poudre de cire de Carnauba. Les échantillons en plexiglas, sous forme de disques aussi, avaient l'épaisseur de 4,5 mm et 29 mm de diamètre.

L'échauffement des échantillons a été fait dans une étuve thermostatée et le mesurement de la charge par induction électrostatique à l'aide d'une électromètre électronique.

En vue de comparer les données expérimentales, nous avons introduit chaque fois dans l'étuve deux échantillons identiques, placés entre électrodes métalliques de nature identique, mais tandis qu'entre les électrodes d'un échantillon on appliquait un champ électrique polarisant, les deux autres étaient court-circuités et mis à la masse. Puis on échauffait jusqu'à 160°C les échantillons en plexiglas et jusqu'à 72°C ceux en cire de Carnauba. On arrivait à la température proposée en 45 minutes environ. Après avoir être maintenus à cette température environ 3 heures, les échantillons étaient refroidis lentement, dans l'étuve, jusqu'à la température ambiante. Le mesurement de la charge rémanente, résultée par la polarisation et la formation de l'électret, a été fait toute suite après la sortie de l'étuve aussi bien qu'après certains intervalles de temps, en vue de déterminer sa variation avec le temps.

Dans la fig. 1 sont présentés les résultats obtenus sur des échantillons en plexiglas et électrodes en bronze, résultats qui confirment qu'ont avait obtenus des électrets dans tous les deux cas, donc aussi dans l'absence du champ électrique polarisant entre les électrodes métalliques.

De la valeur initiale de la charge électrique sur les deux surfaces des électrets tellement obtenus et de sa variation en temps il en résulte que les valeurs des charges initiales, tout pour les échantillons obtenus sous l'action du champ électrique que pour ceux obtenus en absence du champ entre les électrodes court-circuités, sont comparables.

Utilisant un champ polarisant de 4500 V/cm, pour l'échantillon formé en présence du champ a résulté une hétérocharge. Pour tous les deux échantillons le décroissement de la charge en temps suit une loi exponentielle, les valeurs des charges sont très voisines et à fin les deux échantillons restent avec une charge électrique positive. Il est donc assez difficile de distinguer l'électret formé sous l'action du champ électrique de celui obtenu en son absence.

Pour vérifier si la valeur de la température d'échauffement et l'intensité du champ électrique influencent le comportement similaire des

échantillons, nous avons préparé des électrets dans des conditions analogues en utilisant pour l'échantillon formé sur l'action du champ, un champ dont la valeur était de 1100 V/cm. Cette fois-ci les échantillons ont été chauffés jusqu'à une température plus basse, d'environ 150°C seulement.

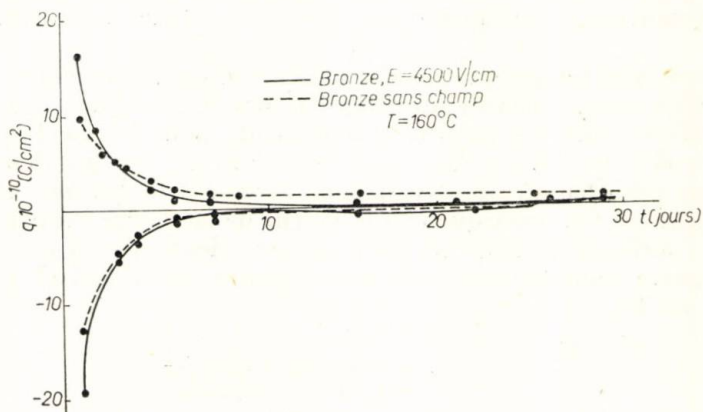


Fig. 1

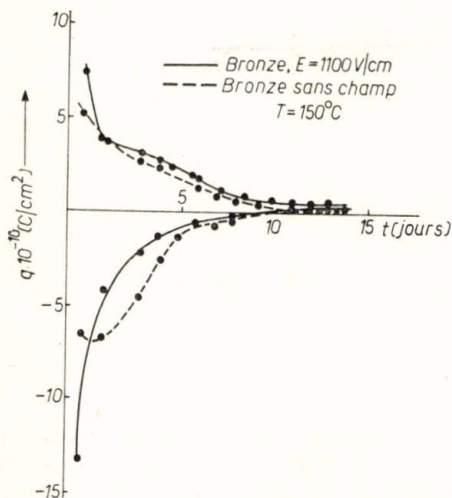


Fig. 2

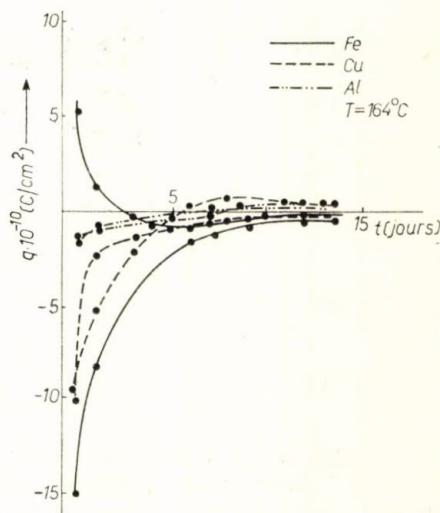


Fig. 3

Les résultats obtenus pour la charge initiale et pour sa variation en temps, données dans la fig. 2, sont en parfait concordance avec ceux obtenus sur les échantillons antérieurs, avec l'observation bien normale que cette fois la valeur de la charge est diminuée.

Pour déterminer comment la nature de l'électrode métallique influence la valeur et la variation de la charge en temps, nous avons préparé des électrets en plexiglas, en absence du champ électrique, en utilisant des électrodes en Cu, Fe, Al.

Les résultats expérimentaux sont illustrés dans la fig. 3. Une polarisation rémanente remarquable est évidente, donc la formation d'un électret.

On voit que tandis que pour les échantillons placés entre électrodes en Fe sur les deux surfaces apparaissent des charges de signe contraire et à la fin des charges négatives seulement, pour les électrets obtenus entre électrodes en Cu ou Al, au moment initial, sur les deux surfaces il n'y a que des charges négatives, pour arriver à la fin aux charges positives seulement. On remarque que la valeur de la charge initiale qui apparaît sur les surfaces de l'électret obtenu entre électrodes en Cu, est, à son tour, beaucoup moindre que celle qui apparaît sur l'électret placé entre électrodes en Fe.

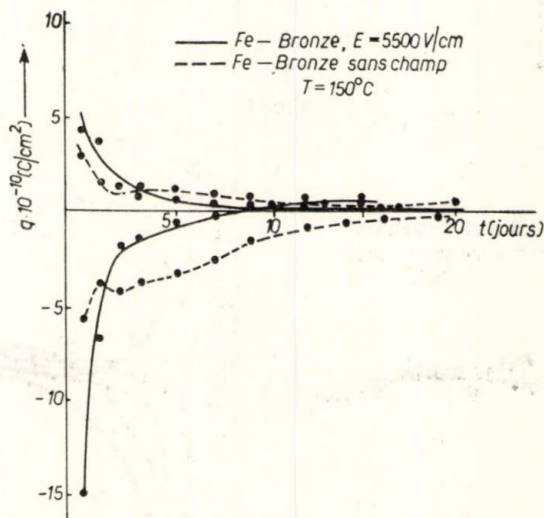


Fig. 4

Dans la fig. 4 on trouve la valeur de la charge initiale et sa variation en temps pour deux électrets en plexiglas, obtenus, un sous l'action du champ électrique et l'autre dans son absence, les deux placés entre deux électrodes métalliques de nature différente (fer et bronze).

Cette fois-ci, les deux électrets ont manifesté le même comportement : sur l'interface de contact avec l'électrode en bronze on constate l'apparition des charges positives, tandis que sur l'interface de contact avec l'électrode en fer on vérifie l'apparition des charges négatives, dont la valeur est supérieure. À la fin, l'électret obtenu dans le champ électrique présente

des charges positives sur les deux surfaces tandis que l'électret obtenu en l'absence du champ, reste avec des charges de signe contraire sur les deux surfaces.

Pour vérifier si les charges qui apparaissent ont une distribution superficielle ou en volume, deux disques en plexiglas ont été mis entre deux électrodes, en bronze et en fer, puis chauffés en absence du champ électrique, obtenant ainsi un électret.

Les résultats expérimentaux concernant la valeur initiale et la variation en temps de la charge sur les quatre surfaces des deux disques qui ont constitué l'échantillon sont donnés dans la fig. 5. Les résultats

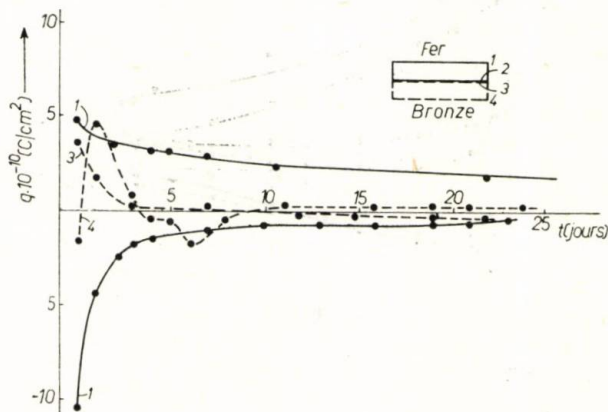


Fig. 5

font évidente, comme l'a d'ailleurs montrée Gubkin aussi [3], une distribution spatiale des charges, irrégulièrement répartie dans la masse du diélectrique.

Pour une complète édification, nous avons préparé aussi, dans les mêmes conditions, c'est à dire en absence du champ électrique, des électrets en cire de Carnauba.

En mettant les échantillons en cire de Carnauba entre électrodes en Fe, Cu ou Al, et les chauffant jusqu'à 72°C, nous avons obtenus des électrets, dont la charge initiale et sa variation en temps ont les valeurs représentées par les courbes de la fig. 6. Cette fois on constate que l'électret formé entre les électrodes en Al a sur les deux surfaces une charge positive, tandis que les électrets formés entre les électrodes en Fe ou en Cu ont sur une surface une charge positive et sur l'autre une charge négative. On peut signaler aussi que dans le cas des électrets en cire de Carnauba la charge positive a une valeur supérieure.

Puisque les comparaisons faites jusqu'à présent étaient rapportées aux électrets obtenu en présence des champs électriques de petite intensité et qui présentaient donc hétérocharges, il a été question de réaliser

des électrets dans des champs plus grandes, 25 KV/cm, qui puissent présenter homocharges.

À ce but ont été préparés des électrets en cire de Carnauba placés entre électrodes en Al ou en Fe, réalisés à la température de 72°C, tant en présence d'un champ électrique de 25 KV/cm qu'en son absence. Les

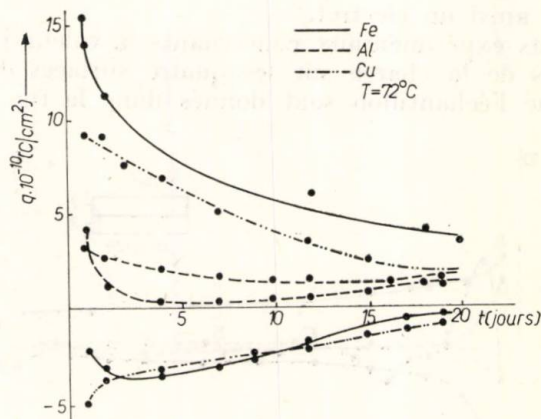


Fig. 6

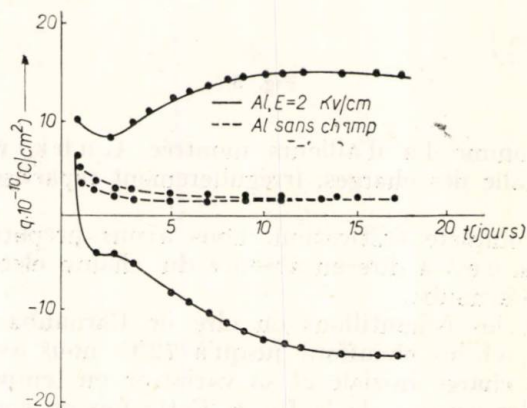


Fig. 7

résultats expérimentaux concernant la valeur initiale et la variation en temps de la charge électrique sont donnés dans les fig. 7 et 8.

De l'analyse des courbes il en résulte pour les électrets obtenus en absence du champ la conclusion d'auparavant : sur les deux surfaces de l'électret formé entre électrodes en Al apparaissent des charges positives, tandis que pour celui formé entre électrodes en Fe il y a deux espèces

de charges, sur une des surfaces des charges positives et sur l'autre — négatives. En ce que concerne les électrets obtenus en présence du champ électrique, les homocharges ont dans les deux cas une tendance initiale

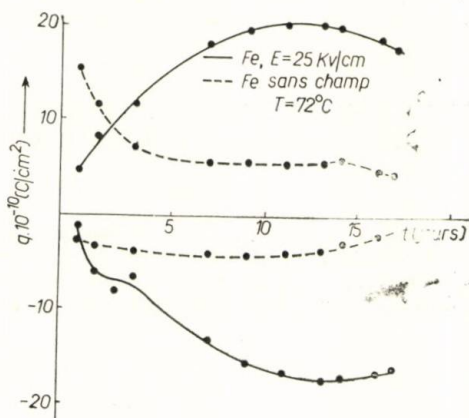


Fig. 8

de croissance, à la seule différence que dans le cas des électrets formé entre électrodes en Al, la charge électrique initiale est positive sur les deux surfaces.

3. Conclusions

1. Fondés sur les résultats expérimentaux, il nous semble évident que dans le processus de formation des électrets les phénomènes présents au contact électret-électrode jouent un rôle très important. Ces phénomènes dépendent de la nature des électrets et des électrodes et peuvent déterminer des processus de polarisation intenses, avec la formation d'un électret, même en absence d'un champ électrique polarisant.

2. L'interprétation des résultats obtenus, respectivement l'élaboration d'une théorie sur le processus intime de polarisation et de formation des électrets, impose premièrement une séparation entre les processus déterminés par l'action du champ électrique polarisant et ceux qui apparaissent au contact métal-diélectrique, séparation qui suppose la considération du potentiel de contact et du travail mécanique d'extraction, c'est à dire encore beaucoup d'autres déterminations expérimentales.

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ASUPRA INFLUENȚEI NATURII ELECTROZILOR METALICI
ÎN PROCESELE DE POLARIZARE ȘI FORMARE A ELECTREȚILOR

(Rezumat)

Se studiază fenomenul de apariție a sarcinii electrice pe fețele unui dielectric încălzit între doi electrozi metalici, în absența unui câmp electric. Se determină mărimea sarcinii și variația acesteia în timp, pentru diferiți electrozi, și se compară cu sarcina electreților obținuți în prezența câmpului electric, în condiții identice de tratament termic.

DÉTERMINATION DES CONSTANTS OPTIQUES DES COUCHES MINCES DE Sb ¹⁾

PAR

SOFIA MELINTE

1. Introduction

La densité et la résistivité des couches minces de Sb ont constitué l'objet d'étude systématique de Cremer et Rueld [1], tandis que leur structure a fait l'objet d'étude de Madeleine Gandais [2]. Les propriétés optiques de ces couches ont été moins étudiées et elles font l'objet de cette recherche.

2. Résultats des recherches

2.1. Obtention des éprouvettes

Les couches de Sb ont été obtenues par la technique courante de la vaporisation thermique sous vide employée par [3]. La pression de l'incinte de vaporisation a été de l'ordre (1,2 ... 3,7) 10^{-6} torr.

2.2. Détermination des constantes optiques

Pour déterminer les constantes optiques des couches minces de Sb on a utilisé une méthode optique en incidence normale (pour éviter une éventuelle anisotropie de la couche) décrite dans [4]. Le résultat des mesures est donné dans les graphiques des fig. 1...8.

Dans la fig. 1 on présente la variation des facteurs R (facteur de réflexion par rapport à l'air), R' (facteur de réflexion par rapport au su-

¹⁾ Présentée lors du Symposium National consacré à l'Interaction de la Radiation avec le Solide, Cluj, avril 1970.

port) et T (facteur de transmission), en fonction de l'épaisseur de la couche de Sb, pour $\lambda = 700 \text{ m}\mu$. On remarque une diminution de la transmission et une augmentation de la réflexion, en fonction de l'épaisseur de la couche. En analysant la fig. 2 on voit que l'énergie effective absorbée par la couche augmente rapidement avec l'épaisseur. Pour une épaisseur de $10,5 \text{ m}\mu$

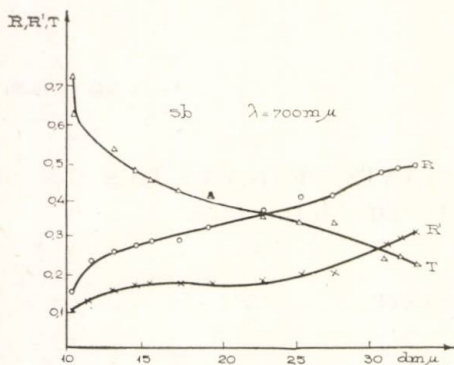


Fig. 1. — Variation des facteurs de réflexion R , R' , et de transmission, T , en fonction de l'épaisseur de la couche de Sb, pour $\lambda = 700 \text{ m}\mu$.

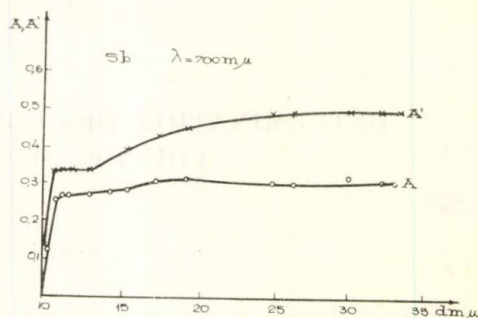


Fig. 2. — Variation des facteurs d'absorption A et A' en fonction de l'épaisseur de la couche de Sb pour un $\lambda = 700 \text{ m}\mu$.

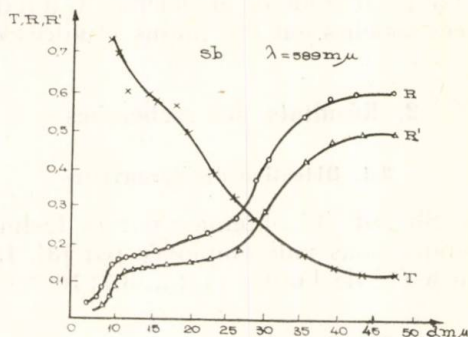


Fig. 3. — Variation des facteurs de réflexion R , R' et de transmission T , en fonction de l'épaisseur de la couche de Sb, pour $\lambda = 589 \text{ m}\mu$.

A atteint un premier maximum mais ensuite l'augmentation est plus lente. Dans le cas d'une longueur d'onde $\lambda = 589 \text{ m}\mu$, les couches pour lesquelles on a déterminé les constantes optiques n'ont pas été obtenues simultanément, fait qu'on remarque aussi dans les graphiques de variation de R , R' et T , avec l'épaisseur.

Dans la fig. 3, on distingue très bien les portions des courbes des facteurs R et R' pour les deux sets de pellicules, (11 et 7). Les énergies absorbées dans les couches de Sb pour $\lambda = 589 \text{ m}\mu$ sont présentées dans la fig. 4. On remarque que le maximum des deux énergies d'absorption (A et A') n'est pas obtenu pour la même épaisseur de la couche. Le maximum de A' est déplacé vers des épaisseurs plus réduites.

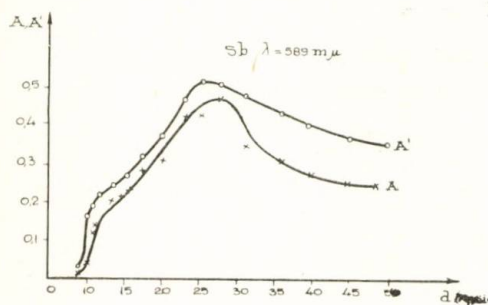


Fig. 4. — Variation des facteurs d'absorption A et A' , en fonction de l'épaisseur de la couche de Sb pour un $\lambda = 589 \text{ m}\mu$.

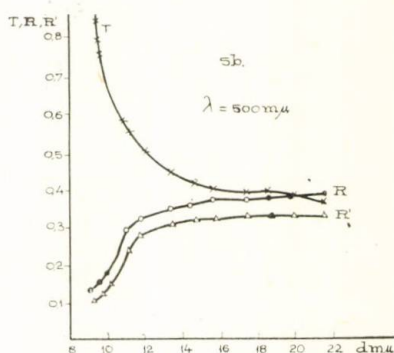


Fig. 5. — Variation des facteurs de réflexion R et R' et de transmission T , en fonction de l'épaisseur de la couche de Sb pour $\lambda = 500 \text{ m}\mu$.

Dans la fig. 5 on constate qu'au cas des couches obtenues simultanément, les facteurs de réflexion et de transmission ne présentent plus des maximums et des minimums; les points correspondants aux valeurs obtenues pour différentes épaisseurs sont situés justement sur la courbe

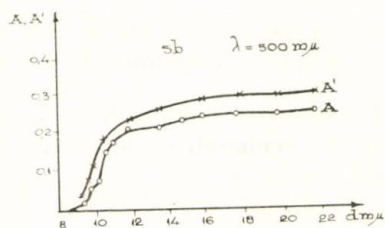


Fig. 6. — Variation des facteurs d'absorption A et A' en fonction de l'épaisseur de la couche de Sb pour $\lambda = 500 \text{ m}\mu$.

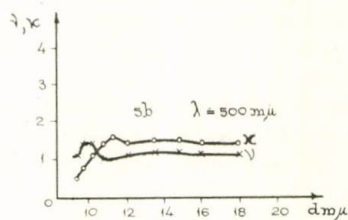


Fig. 7. — Variation des indices de réfraction ν et d'extinction κ en fonction de l'épaisseur de la couche de Sb pour $\lambda = 500 \text{ m}\mu$.

La fig. 6 met en évidence une augmentation rapide de l'absorption jusqu'à $d = 12 \text{ m}\mu$, suivie d'une augmentation de beaucoup plus lente.

Dans les fig. 7 et 8 on donne les variations des indices de réfraction ν et d'extinction κ en fonction de l'épaisseur de la couche pour un

$\lambda = 500 \text{ m}\mu$ et $589 \text{ m}\mu$. On voit que ν a une valeur maximale pour d compris entre $8,9$ et $10,2 \text{ m}\mu$, puis baisse vers la valeur 1 et pour des épaisseurs plus réduites oscille autour de cette valeur. L'extinction augmente avec l'accroissement de l'épaisseur et l'on constate que le maximum de l'extinction coïncide avec le maximum de l'indice de refraction

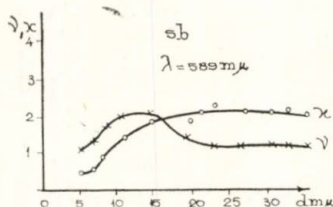


Fig. 8. — Variation des indices de réfraction ν et d'extinction χ en fonction de l'épaisseur de la couche de Sb pour un $\lambda = 589 \text{ m}\mu$.

Dans la fig. 8 on remarque que la maximum de ν est obtenu pour un domaine plus large d'épaisseurs. L'extinction accroit avec l'épaisseur, atteignant la valeur maximale pour $d = 23 \text{ m}\mu$ pour rester ensuite constante.

3. Conclusions

1. Pour avoir des résultats dignes d'être reproduits il faut obtenir les couches de manière simultanée, dans les mêmes conditions de travail (pression, vitesse des dépôts etc).

2. Les facteurs de transmission et de réflexion dépendent de la longueur d'onde avec laquelle on fait les recherches, ainsi que de l'épaisseur de la couche.

3. Les indices de réfraction ν et d'extinction χ dépendent de la longueur d'onde avec laquelle on effectue les recherches, de l'élément utilisé et de son épaisseur dans les couches minces.

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DETERMINAREA CONSTANTELOR OPTICE ALE STRATURILOR SUBȚIRI DE Sb

(Rezumat)

Se studiază proprietățile optice ale straturilor subțiri de Sb în domeniul vizibil. Grosimea straturilor s-a determinat printr-o metodă optică în incidență normală. S-a lucrat în domeniul grosimilor $5 \text{ m}\mu < d < 35 \text{ m}\mu$. Straturile au fost obținute prin evaporare termică sub vid, vidul fiind de ordinul $2,6 \cdot 10^{-6}$ torr.

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EINFLUSS DER BEHANDLUNGSTEMPERATUR AUF DIE PLASTISCHE DEFORMIERTE TEXTURLEGIERUNG

Fe—3,27% Si ¹⁾

VON

EMIL LUCA, GH. CĂLUGĂRU und ELENA DIACONU

1. Einleitung

Die Texturentwicklung oder die Vorzugsorientierung in den plastisch kaltdeformierten und wiederkristallisierten Metallen, ist den Metallurgen wohl bekannt. Als Folge dieser Texturen können die ferromagnetischen Metallen eine magnetische Anisotropie aufweisen [1].

Die Legierung Fe—Si im kaltgewalzten Zustand bietet eine Deformierungstextur, die keine praktische Anwendung findet und nur als eine Stufe zur thermischen Texturbehandlung wichtig ist [2].

In Abhängigkeit von Behandlungstemperatur kann die kaltgewalzene Legierung Fe—Si den Glühen- oder Wiederkristallisierungsprozess leiden.

In der vorliegenden Arbeit, durch die Verwendung des Verfahren der Torsionkurven untersucht man den Einfluss der Behandlungstemperaturen auf die Textur der Legierung Fe — 3,27% Si und die Bestimmung der Übergangstemperatur vom Glühen zur Wiederkristallisierung. Ebenfalls wird die durch das magnetische Glühen der Legierung Fe—Si, induzierte Anisotropie untersucht.

2. Experimentelles Vorgehen

Die Darstellung der Torsionkurven wurde mit einem Fadenmagnetometer auf 0,35 mm dicken und mit 2 cm² Oberfläche Versuchsscheiben durchgeführt.

¹⁾ Mitgeteilt an der wissenschaftlichen Tagung der Fakultät für Elektrotechnik Jassy, am 21. — 23. Dezember 1972.

Die Legierung Fe—Si mit 3,27% Si enthält auch folgende Unreinigungen: C = 0,02%, Mn = 0,06%, P = 0,012%, S = 0,004%, Al = 0,011%.

Man hat mit zwei Serien von Versuchsscheiben gearbeitet:

I Serie — Versuchsscheiben die wie folgt behandelt wurden: warm gewalzen in Bänder von $2,5 \times 7,5$ mm; 4 Stunden lang warmbehandelt bei 950°C, dann folgt ein langsames Kühlen, und dann wieder kalt gewalzen in Band von 0,35 mm Dicke. Die in den Versuchen verwendeten Scheiben wurden aus diesen Bänder herausgeschnitten.

II Serie — Versuchsscheiben die, nachdem sie wie die erste Serie behandelt wurden, nach 6 Stunden lang einer Behandlung im luftleeren Raum von $3 \cdot 10^{-3}$ torr bei 1200°C unterzogen wurden, alles gefolgt von einem langsamen Kühlen.

3. Experimentelle Ergebnisse und Diskussionen

Die Torsionskurve der kaltgewalzenen Fe—3,27% Si Versuchsscheiben (*I* Serie) ist von Periode π also bietet eine einachsige magnetische Anisotropie mit einer leichtmagnetischen Achse in der Walzrichtung (Abb. 1, Kurve 1).

Diese Feststellung ist im Einklang mit Rathenaus und Snoeks [3] Ergebnissen. Diese haben dieselbe Richtung der leicht magnetischen Achse bei der polykristallin gewalzene Legierung Ni_3Fe gefunden. Diese Art von Anisotropie wurde von Chikazumi [4] durch Betrachtung der Richtungsordnung, die sich bei einer Rutschenformänderung induziert, erklärt.

Unsere Ergebnisse für gewalzenen Fe—Si stehen im gegensatz mit denen Tarasovs [5] der bei den gewalzenen Fe—Si eine stark monokomponente Textur findet, ähnlich mit der Textur wiederkristallisierten Legierung. Eine gleichartige Textur wurde von uns bei den wiederkristallisierten Versuchsscheiben gefunden. Die Kurve 2 stellt im Abb. 1 die Torsionskurve für die wiederkristallisierten Proben Fe—Si (Serie *II*) dar. Durch die Wiederkristallisierung erhalten die Proben eine stark monokomponente Richtungstextur $\{110\} <100>$.

Weil die Textur der kaltgewalzenen Probe unverändert ist [2], kann die magnetische Anisotropie, um das Glühen von der Wiederkristallisierung zu unterscheiden, verwendet werden. Diese Technik wurde von Conradt und von seinen Mitarbeitern [6] auf Ni und auf die Legierung Fe—Ni angewendet.

Um die Anfangstemperatur der Wiederkristallisierung bestimmen zu können werden Torsionskurven der ersten Serie von Versuchsscheiben (kaltgewalzen) und die Kurven derselben Serie aber nach einer Behandlung von zwei Stunden bei verschiedenen Temperaturen dargestellt (Abb. 2).

Unsere Forschungen zeigen dass die Form der Torsionskurven, also die Walztextur sich nicht im Falle der Behandlung bei 400°C oder 500°C

verändern. Das Steigen der Behandlungstemperatur über 500°C führt zu einer stufenweiser Veränderung der Anisotropie, also zu einer immer akzentuierteren Wiederkristallisierung.

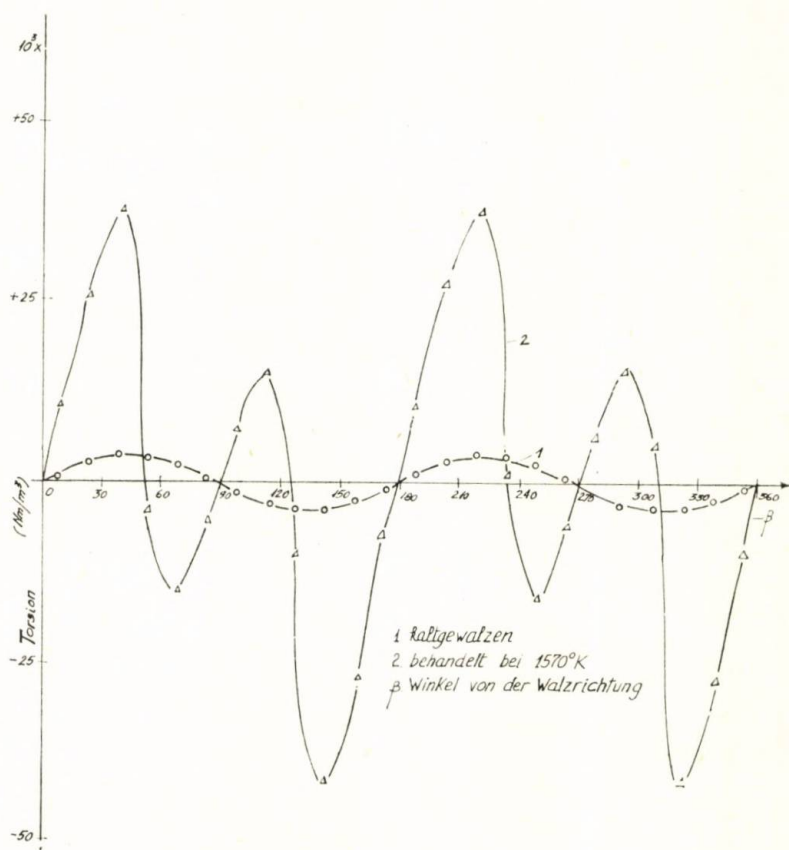


Abb. 1

Da die Wiederkristallisierung bei Temperaturen von über 500°C stattfindet, haben wir die in Fe—Si durch thermomagnetische Behandlung induzierte Anisotropie auf Versuchsscheiben die, bei einer Temperatur von 400°C und 500°C in einem Feld von $4 \cdot 10^4$ A/m behandelt wurden, untersucht.

Es werden die Torsionkurven vor und nach der thermomagnetischen Behandlung dargestellt und aus ihrer Differenz wurde die Kurve von Periode π die der induzierten Anisotropie entspricht, gefunden.

Die experimentellen Werte für K_u sind von der Ordnung $K_u = 3\,300 \text{ J/m}^3$ bei 400°C und $K_u = 1\,300 \text{ J/m}^3$ bei 500°C . Als Vergleichsmittel wurden auch die theoretischen Werte mit der Richtungsordnenden Theorie

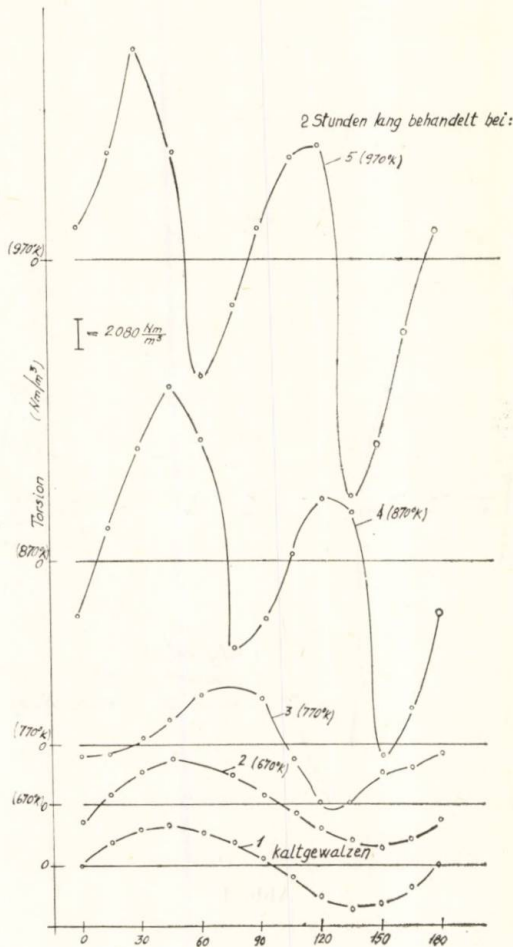


Abb. 2

von Néel-Taniguchi ausgerechnet [7]...[9]. Bei der Berechnung der Konstante der induzierten Anisotropie K_u wurde, der von Moses und Thompson [10] modifizierte Ausdruck Taniguchis:

$$K_u = \frac{A N n^2 C_1 C_0 I_s(T)^2 I_s(T')^2}{k T' I_s(0)^4}$$

verwendet, wobei A eine von der Netzstruktur und von der Feldrichtung während der Behandlung abhängige Konstante ist, N die Atomzahl auf die Volumeneinheit, n die Konzentration der gelosten Atome, k die Boltzmann Konstante, T' die Behandlungstemperatur, $I_s(T)$ der Wert der Sättigungsmagnetisierung bei T (gleichartig für T' und 0°K) ist.

Wir erwähnen dass die Werte der Konstanten A , C_1 , C_0 aus [10] entnommen wurden.

Die in den gemachten Annäherungen gefundene theoretische Werte sind: bei 400°C , $K_u = 900 \text{ J/m}^3$ und bei 500°C , $K_u = 700 \text{ J/m}^3$.

Man bemerkt dass diese Werte, im Vergleich mit den experimentell erhaltenen Werte, kleiner sind. Sie sind aber im Einklang mit den Ergebnissen Moses und Thompson [10] die bei einer Behandlungstemperatur von 700°C für K_u den Wert $K_u = 680 \text{ J/m}^3$, also auch über den theoretischen Wert gefunden haben.

4. Schlussfolgerungen

1. Die kaltgewalzene Legierung Fe—Si mit 3,27% Si hat eine einachsige Anisotropie von leichter Magnetisierung in die Richtung der Walzung.

2. Das Phänomen der Wiederkristallisierung erscheint bei einer Temperatur von über 500°C . Die Wiederkristallisierung nimmt mit der Behandlungstemperatur zu.

3. Die durch thermomagnetische Behandlung induzierte Anisotropie K_u in der gewalzenen Legierung Fe — 3,27% Si hat einen grösseren Wert als der mit der Theorie von Néel-Taniguchi theoretische berechnete Wert.

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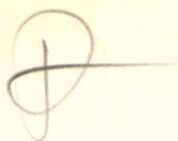
INFLUENȚA TEMPERATURII DE TRATAMENT ASUPRA TEXTURII
ALIAJULUI Fe—3,27% Si DEFORMAT PLASTIC

(Rezumat)

Se studiază, prin metoda curbelor de torsiune, influența temperaturii de tratament asupra texturii aliajului Fe — 3,27% Si și se determină temperatura de trecere de la recoacere la recristalizare. De asemenea se cercetează anizotropia magnetică indusă prin recoacerea magnetică a aliajului Fe—Si.

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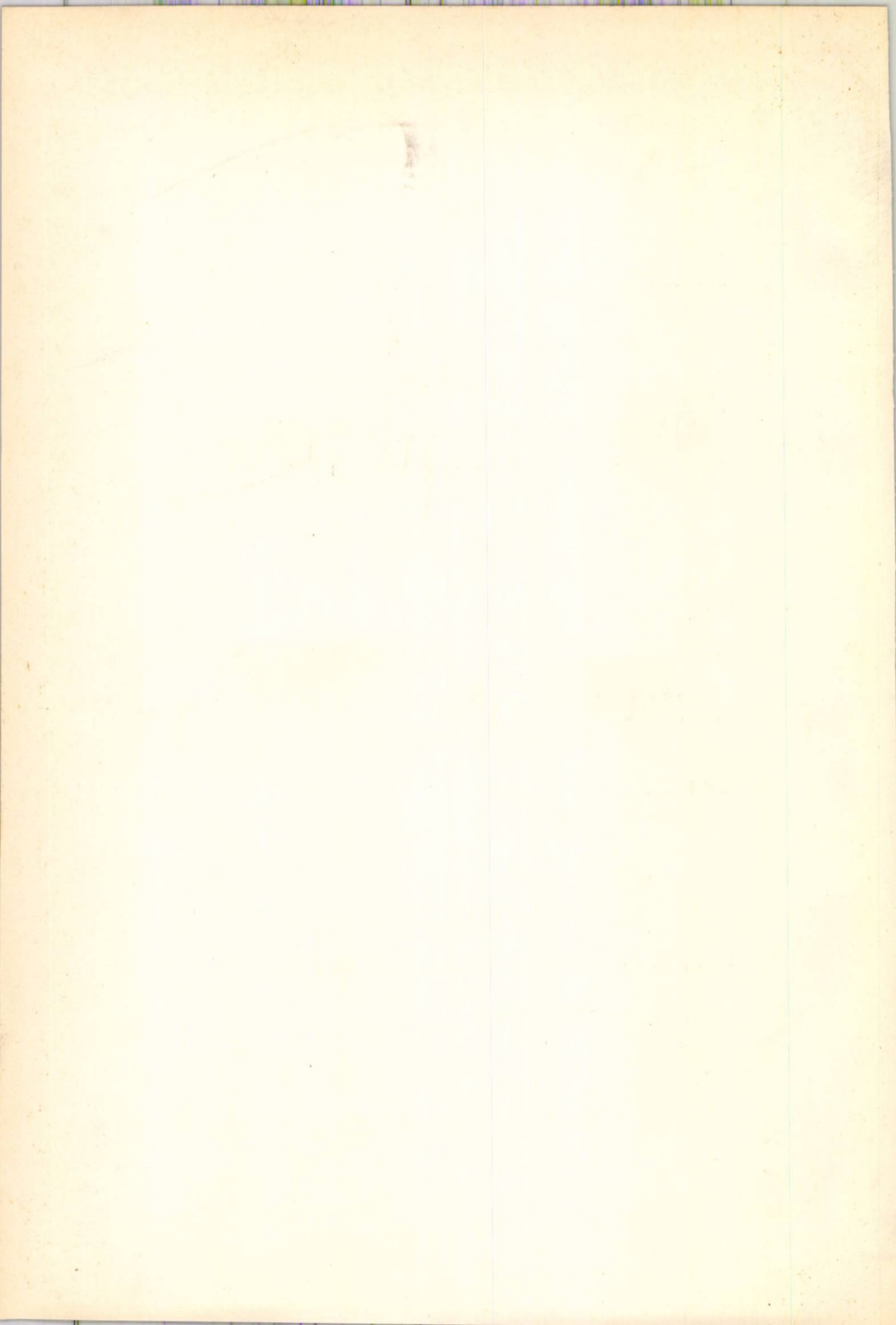
TOMUL XIX (XXIII)

Fasc. 3—4



**SECTIA I MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1973



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ASUPRA GRUPULUI UNOR CLASE DE DRUMURI ÎNTR-UN SPAȚIU TOPOLOGIC CU SISTEM LOCAL DE Δ -MODULE¹⁾

DE

I. POP

1. În această Notă se consideră categoria $\mathcal{O}$ a spațiilor topologice cu sisteme locale de Δ -module, se definesc drumurile într-un obiect $(X, \Gamma) \in \text{Ob } \mathcal{O}$ ca fiind morfisme de la (I, Δ) la (X, Γ) , se consideră drumurile φ_0 -închise în $x_0 \in X$ și se arată că, clasele de omotopie, în $\mathcal{O}$, ale acestor drumuri formează un grup $\pi(X, x_0, \Gamma, \varphi_0)$. Se stabilesc unele proprietăți ale acestui grup. Se consideră proiecții de acoperire în categoria $\mathcal{O}$ și se stabilește legătura cu grupurile atașate spațiilor bază și total.

2. Fie Δ un inel comutativ cu unitate. Vom nota prin $\mathcal{O}$ categoria spațiilor topologice cu sisteme locale de Δ -module. Prin urmare [1] $(X, \Gamma) \in \text{Ob } \mathcal{O}$ dacă $X \in \text{Ob Top}$ și $\Gamma: P(X) \rightarrow \Delta\text{-mod}$ este un functor covariant, de la grupoidul fundamental al spațiului X la categoria Δ -modulelor. Amintim de asemenea că un morfism $f: (X, \Gamma) \rightarrow (Y, \Gamma')$ în categoria $\mathcal{O}$, este format dintr-o aplicație continuă $f: X \rightarrow Y$ și, pentru fiecare $x \in X$ un morfism de module $f_x: \Gamma(x) \rightarrow \Gamma'(f(x))$ astfel că dacă ω este un drum în X să avem:

$$(1) \quad f_{\omega(1)} \circ \Gamma(\omega) = \Gamma'(f \circ \omega) \circ f_{\omega(0)}.$$

Vom considera $I = [0, 1]$ cu topologia de subspațiu al axei reale și cu sistemul local constant $\Gamma(t) = \Delta$ și $\Gamma(\theta) = 1_\Delta$.

Fie $(X, \Gamma) \in \text{Ob } \mathcal{O}$. Vom numi *drum în* (X, Γ) orice morfism $(I, \Delta) \rightarrow (X, \Gamma)$, în categoria $\mathcal{O}$. Prin urmare, un drum în (X, Γ) este definit printr-un drum $\omega: I \rightarrow X$ în categoria Top și, pentru orice $t \in I$ un omomor-

¹⁾ Prezentată la sesiunea științifică a Universității „Al. I. Cuza” din Iași, 26—29 octombrie 1972.

fism de Δ -module $\omega_t : \Delta \rightarrow \Gamma(\omega(t))$ cu condiția că dacă $\theta : I \rightarrow I$ este o aplicație continuă, să avem :

$$(2) \quad \Gamma(\omega \circ \theta) \circ \omega_{\theta(0)} = \omega_{\theta(1)}.$$

Dacă în particular $\theta = 1_I$ obținem

$$(3) \quad \Gamma(\omega) \circ \omega_0 = \omega_1,$$

iar dacă considerăm $\theta_t(\tau) = t\tau$, atunci din formula (2) se obține :

$$(4) \quad \Gamma(\omega \circ \theta_t) \circ \omega_0 = \omega_t.$$

Rezultă din (4) că un drum în (X, Γ) este complet determinat de aplicația continuă ω și omomorfismul ω_0 . Nu rezultă însă că orice pereche (ω, ω_0) este un drum în (X, Γ) deoarece acestea trebuie să fie astfel încît :

$$(5) \quad \Gamma(\omega \circ \theta \times \omega \circ \theta_{\theta(0)}) \circ \omega_0 = \Gamma(\omega \circ \theta_{\omega(1)}) \circ \omega_0,$$

pentru orice $\theta : I \rightarrow I$ continuă.

Exemple de drumuri în (X, Γ) . 1. Fiecărui drum ω în X , în categoria Top, îi putem atașa un drum în (X, Γ) luînd $\omega_t = 0$, pentru orice $t \in I$.

2. Dacă (ω, ω_t) este un drum în (X, Γ) și $\theta : I \rightarrow I$ este o aplicație continuă atunci putem considera omomorfismul $(\omega \circ \theta)_t = \omega_{\theta(t)}$, pentru orice $t \in I$. Rezultă imediat că $(\omega \circ \theta, (\omega \circ \theta)_t)$ este un drum în (X, Γ) .

3. Fie $(X, \Gamma) \in \text{Ob } \mathcal{C}$ și $\varphi_0 : \Delta \rightarrow \Gamma(x_0)$ un omomorfism de Δ -module. Dacă $\varepsilon : I \rightarrow X$ este drumul constant $\varepsilon(t) = x_0$ și luăm $\varepsilon_t = \varphi_0$ pentru orice $t \in I$, atunci $(\varepsilon, \varepsilon_t) = (x_0, \varphi_0)$ este un drum în (X, Γ) numit drum constant.

4. Dacă (ω, ω_t) este un drum în (X, Γ) atunci $(\hat{\omega}, \hat{\omega}_t)$ cu $\omega(t) = \omega(1-t)$ și $\hat{\omega}_t = \omega_{1-t}$ este de asemenea un drum în (X, Γ) .

5. Fie $(\omega, \omega_t), (\omega', \omega'_t)$ două drumuri în (X, Γ) cu $\omega(1) = \omega'(0)$ și $\omega_1 = \omega'_0$. Vom considera atunci $(\omega \times \omega', (\omega \times \omega')_t)$ cu $\omega \times \omega'$ compunerea drumurilor ω și ω' în categoria Top și

$$(\omega \times \omega')_t = \begin{cases} \omega_{2t}, & 0 \leq t \leq \frac{1}{2} \\ \omega'_{2t-1}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Propoziția 1. *Perechea $(\omega \times \omega', (\omega \times \omega')_t)$ este un drum în (X, Γ) , numit compunerea drumurilor (ω, ω_t) și (ω', ω'_t) .*

Demonstrație. Fie $\theta : I \rightarrow I$ cu $\theta(0) = t_0$ și $\theta(1) = t_1$. Avem $\theta \simeq \theta_1$ rel I unde $\theta_1(t) = (1-t)t_0 + t t_1$, prin omotopia $F(t, \tau) = (1-\tau)\theta(t) + \tau\theta_1(t)$. Dar atunci $(\omega \times \omega') \circ \theta \simeq (\omega \times \omega') \circ \theta_1$ rel I . Avem prin urmare $\Gamma((\omega \times \omega') \circ \theta) = \Gamma((\omega \times \omega') \circ \theta_1)$. Dacă $t_0, t_1 \leq 1/2$ sau $t_0, t_1 \geq 1/2$, atunci se verifică imediat formula (2).

Să presupunem că $t_0 < 1/2$ și $t_1 > 1/2$. Arătăm că are loc

$$(6) \quad (\omega \times \omega') \circ \theta_1 \simeq (\omega \circ \bar{\theta}) \times (\omega' \circ \bar{\theta}) \text{ rel } I,$$

unde $\bar{\theta}(t) = (1 - 2t_0)t + 2t_0$ și $\bar{\theta}(t) = (2t_1 - 1)t$. Fie pentru aceasta aplicația continuă $\varphi: I \rightarrow I$ definită prin

$$\varphi(t) = \begin{cases} \frac{(1 - 2t_0)t}{t_1 - t_0}, & t \leq \frac{1}{2} \\ \frac{2t_1 - t_1 - t - t_0 + 1}{t_1 - t_0}, & t \geq \frac{1}{2} \end{cases}.$$

Se constată ușor că $((\omega \times \omega') \circ \theta_1) \circ \varphi = (\omega \circ \bar{\theta}) \times (\omega' \circ \bar{\theta})$ și cum $\varphi(0) = 0$ și $\varphi(1) = 1$ rezultă [2] că are loc (6). Avem acum $\Gamma((\omega \times \omega') \circ \theta_1) \circ (\omega \times \omega')_{\theta_1(0)} = \Gamma(\omega' \circ \bar{\theta}) \circ \Gamma(\omega \circ \bar{\theta})_{\omega_{2t_0}} = \Gamma(\omega' \circ \bar{\theta}) \Gamma(\omega \circ \bar{\theta}) \omega_{\bar{\theta}(0)} = \Gamma(\omega' \circ \bar{\theta}) \omega_{\bar{\theta}(1)} = \Gamma(\omega' \circ \bar{\theta}) \omega'_0 = \Gamma(\omega' \circ \bar{\theta}) \omega'_{\bar{\theta}(0)} = \omega'_{\bar{\theta}(1)} = \omega'_{2t_1-1} = (\omega \times \omega')_{\theta_1(1)}$.

Cu aceasta propoziția este complet demonstrată.

Fie $(X, \Gamma) \in \text{Ob } \mathcal{O}$ și $\varphi_0: \Delta \rightarrow \Gamma(x_0)$ un omomorfism de Δ -module, $(x_0 \in X)$. Un drum în (X, Γ) , (ω, ω_t) , cu $\omega(0) = \omega(1) = x_0$ și $\omega_0 = \omega_1 = \varphi_0$ se numește φ_0 -închis în x_0 .

Vom nota mulțimea drumurilor φ_0 -închise în x_0 în (X, Γ) prin $\Omega(X, x_0, \Gamma, \varphi_0)$. Această mulțime conține drumul constant (x_0, φ_0) și este închisă în raport cu operația $(\omega, \omega_t) \rightarrow (\hat{\omega}, \hat{\omega}_t)$ și cu operația de compunere.

Introducem în mulțimea $\Omega(X, x_0, \Gamma, \varphi_0)$ o relație de echivalență. Două drumuri $(\omega, \omega_t), (\omega', \omega'_t) \in \Omega(X, x_0, \Gamma, \varphi_0)$ le vom numi *echivalente omotopice în $\mathcal{O}$* dacă există o omotopie în Top , $F: \omega \simeq \omega'$ rel I și pentru orice $(t, t') \in I \times I$ un omomorfism $F_{(t,t')}: \Delta \rightarrow \Gamma(F(t, t'))$, care satisface condițiile:

$$a) F_{(t,0)} = \omega_t, F_{(t,1)} = \omega'_t,$$

$$b) F_{(0,t')} = F_{(1,t')} = \varphi_0,$$

c) dacă $\theta: I \rightarrow I$ este o aplicație continuă atunci are loc relația $\Gamma(F_t \circ \theta) \circ F_{(t,\theta(0))} = F_{(t,\theta(1))}$, unde $F_t(t') = F(t, t')$. Se verifică ușor că se obține o relație de echivalență algebrică pe mulțimea $\Omega(X, x_0, \Gamma, \varphi_0)$. Vom nota clasa de echivalență a unui drum $(\omega, \omega_t) \in \Omega(X, x_0, \Gamma, \varphi_0)$ prin $[(\omega, \omega_t)]_{(\Gamma, \varphi_0)}$, iar mulțimea acestor clase de echivalență prin $\pi(X, x_0, \Gamma, \varphi_0)$.

Propoziția 2. *Mulțimea $\pi(X, x_0, \Gamma, \varphi_0)$ are o structură de grup.*

Demonstrație. Se poate arăta ușor că dacă $(\omega, \omega_t) \simeq (\omega', \omega'_t)$ și $(\theta, \theta_t) \simeq (\theta', \theta'_t)$ atunci $(\omega \times \theta, (\omega \times \theta)_t) \simeq (\omega' \times \theta', (\omega' \times \theta')_t)$ adică putem defini compunerea $[(\omega, \omega_t)]_{(\Gamma, \varphi_0)} [(\omega', \omega'_t)]_{(\Gamma, \varphi_0)} = [(\omega \times \omega'), (\omega \times \omega')_t]_{(\Gamma, \varphi_0)}$. Demonstrația propoziției este cu totul analogă cazului grupului fundamental, așa încît o omitem.

Exemple. 1. Dacă $\varphi_0 = 0 : \Delta \rightarrow \Gamma(x_0)$ și $(\omega, \omega_t) \in \Omega(X, x_0, \Gamma, 0)$, atunci rezultă din formula (4) $\omega_t = 0$ și obținem imediat că $\pi(X, x_0, \Gamma, 0) = \pi(X, x_0)$, pentru orice sistem local de Δ -module Γ pe X .

2. Fie M un Δ -modul și $\Gamma(x) = M$, $\Gamma(\omega) = 1_M$, pentru orice $x \in X$. Rezultă atunci tot din (4) că $\omega_t = \omega_0 = \varphi_0$. Se constată ușor acum că $\pi(X, x_0, M, \varphi_0) \approx \pi(X, x_0)$.

Fie $(X, x_0, \Gamma, \varphi_0)$ și $(X, x_1, \Gamma, \varphi_1)$. Dacă θ este un drum în (X, Γ) cu $\theta(0) = x_0$, $\theta(1) = x_1$ și $\theta_0 = \varphi_0$, $\theta_1 = \varphi_1$ atunci $h_\theta[(\omega, \omega_t)]_{(\Gamma, \varphi_0)} = [\theta \times \omega \times \theta^{-1}, (\theta \times \omega \times \theta^{-1})_t]_{(\Gamma, \varphi_0)}$ dă un izomorfism al grupurilor $\pi(X, x_0, \Gamma, \varphi_0)$ și $\pi(X, x_1, \Gamma, \varphi_1)$.

3. Vom considera acum categoria $\mathcal{O}'$ ale cărei obiecte sînt tripletele (X, Γ, φ) unde $(X, \Gamma) \in \text{Ob } \mathcal{O}$ iar $\varphi = (1_X, \varphi) : (X, \Delta) \rightarrow (X, \Gamma)$ este un morfism în $\mathcal{O}$. Un morfism $f : (X, \Gamma, \varphi) \rightarrow (Y, \Gamma', \Psi)$ este un morfism $f : (X, \Gamma) \rightarrow (Y, \Gamma')$, în $\mathcal{O}$, care satisface condiția :

$$(7) \quad f_x \varphi_x = \Psi_{f(x)}.$$

Dacă $f : (X, x_0, \Gamma, \varphi) \rightarrow (Y, y_0, \Gamma', \Psi)$ este un morfism în categoria $\mathcal{O}'$, atunci avem

$$f_\Omega : \Omega(X, x_0, \Gamma, \varphi_x) \rightarrow \Omega(Y, y_0, \Gamma', \Psi_{y_0})$$

cu $f_\Omega(\omega, \omega_t) = (f\omega, f_{\omega(t)}\omega_t)$. Această aplicație induce omomorfismul

$$(8) \quad f_* : \pi(X, x_0, \Gamma, \varphi_x) \rightarrow \pi(Y, y_0, \Gamma', \Psi_{y_0}).$$

În particular, dacă (X, Γ, φ) este un obiect în categoria $\mathcal{O}'$ atunci $\varphi : (X, \Delta, x_0, 1_\Delta) \rightarrow (X, x_0, \Gamma, \varphi_x)$ este un morfism în $\mathcal{O}'$ și acesta induce monomorfismul

$$(9) \quad \varphi_* : \pi(X, x_0, \Delta, 1_\Delta) \rightarrow \pi(X, x_0, \Gamma, \varphi_x).$$

În baza unui rezultat anterior obținem :

Corolarul 1. Dacă (X, Γ, φ) este un obiect în categoria $\mathcal{O}'$ atunci există un monomorfism $\varphi_* : \pi(X, x_0) \rightarrow \pi(X, x_0, \Gamma, \varphi_x)$, definit prin $\varphi_*[(\omega)] = [(\omega, \varphi_{\omega(t)}\varphi_{x_0})]_{(\Gamma, \varphi_x)}$.

Un obiect (X, Γ, φ) îl vom numi *liniar conex în $\mathcal{O}'$* dacă pentru oricare două puncte $x_0, x_1 \in X$ există un drum (ω, ω_t) în $\mathcal{O}$ încît $\omega(0) = x_0$, $\omega(1) = x_1$, $\omega_0 = \varphi_{x_0}$ și $\omega_1 = \varphi_{x_1}$. Dacă (X, Γ, φ) este liniar conex în $\mathcal{O}'$, atunci grupurile $\pi(X, x, \Gamma, \varphi_x)$ sînt izomorfe oricare ar fi $x \in X$. Vom nota în acest caz $\pi(X, \Gamma, \varphi)$.

Fie $f, g : (X, \Gamma, x_0, \varphi) \rightarrow (Y, \Gamma', y_0, \Psi)$ două morfisme în categoria $\mathcal{O}'$, cu $f_{x_0} = g_{x_0}$. Vom spune că f este echivalent omotopic cu g în $\mathcal{O}'$ [1], dacă $F : f \simeq g$ în Top și pentru orice $(x, t) \in X \times I$ există un omomorfism $F_{(x,t)} : \Gamma(x) \rightarrow \Gamma'(F(x, t))$ satisfăcînd condițiile:

$$a') \quad F_{(x,0)} = f_x, F_{(x,1)} = g_x, F_{(x_0,t)} = f_{x_0} = g_{x_0},$$

$$b') \quad \Gamma'(F_x \circ \theta) F_{(x,\theta(0))} = F_{(x,\theta(1))}, \text{ pentru orice } \theta : I \rightarrow I, \text{ unde } F_x(t) = F(x, t),$$

$$c') \quad F_{(x,t)} \varphi_x = \Psi_{F(x,t)}.$$

Aceasta este o relație de echivalență și dacă $f \simeq g$ în $\mathcal{C}'$ atunci $f_* = g_* : \pi(X, x_0, \Gamma, \varphi) \rightarrow \pi(Y, y_0, \Gamma', \Psi)$. Așadar avem :

Corolarul 2. Dacă (X, Γ, φ) și (Y, Γ', Ψ) sînt două obiecte liniar conexe și echivalente omotopic în $\mathcal{C}'$, atunci grupurile $\pi(X, \Gamma, \varphi)$ și $\pi(Y, \Gamma', \Psi)$ sînt izomorfe.

Un obiect (X, Γ, φ) este contractibil în $\mathcal{C}'$ dacă este echivalent omotopic (în $\mathcal{C}'$) cu un obiect $(\{x_0\}, \Gamma(x_0), \varphi_{x_0})$.

Lema 1. Oricare ar fi obiectul (R, Γ, φ) el este contractibil în categoria $\mathcal{C}'$.

Demonstrație. Fie $f : (R, \Gamma) \rightarrow (\{r_0\}, \Gamma(r_0))$, $r_0 \in R$, morfismul în $\mathcal{C}$ definit prin $f(r) = r_0$ și $f_r = \Gamma(\omega)$ unde $\omega(t') = (1 - t')r + t'r_0$. În adevăr, dacă $\theta : I \rightarrow R$, fie $\omega_1(t') = (1 - t')\theta(1) + t'r_0$ și $\omega_2(t') = (1 - t')\theta(0) + t'r_0$. Avem atunci $\theta \times \omega_1 \simeq \omega_2$ și prin urmare $\Gamma(\theta \times \omega_1) = \Gamma(\omega_2)$ adică $\Gamma(\omega_1) \Gamma(\theta) = \Gamma(\omega_2)$ care este echivalentă cu relația $f_{\theta(1)} \Gamma(\theta) = f_{\theta(0)}$, adică f este morfism în categoria $\mathcal{C}$. Să considerăm acum morfismul $i : (\{r_0\}, \Gamma(r_0)) \rightarrow (R, \Gamma)$ cu $i_{r_0} : \Gamma(r_0) \rightarrow \Gamma(r_0)$ identitatea. Morfismul $i \circ f : (R, \Gamma) \rightarrow (\{r_0\}, \Gamma(r_0))$ este echivalent omotopic în $\mathcal{C}$ cu morfismul identitate. În adevăr, fie $F : R \times I \rightarrow R$ cu $F(r, t) = tr_0 + (1 - t)r$ și $F_{(r,t)} : \Gamma(r) \rightarrow \Gamma(F(r, t))$ omomorfismul $F_{(r,t)} = \Gamma(\bar{\omega})$ unde $\bar{\omega}(t') = (1 - t')r_0 + t'(tr_0 + (1 - t)r)$. Se verifică ușor condițiile $a')$, $b')$ și $c')$.

Din corolarul 2 și lema 1 obținem :

Corolarul 3. Pentru orice obiect $(R, \Gamma, \varphi) \in \text{Ob } \mathcal{C}'$ avem $\pi(R, \Gamma, \varphi) = 0$ (grupul trivial).

4. Fie $p : \tilde{X} \rightarrow X$ o proiecție de acoperire și $(\tilde{X}, \tilde{\Gamma}) \in \text{Ob } \mathcal{C}$. Vom arăta că există un sistem local de Δ -module pe X astfel încît p devine un morfism în categoria $\mathcal{C}$.

Vom defini, pentru orice $x \in X$, $\Gamma(x) = \prod_{\tilde{x} \in p^{-1}(x)} \tilde{\Gamma}(\tilde{x})$ (produs direct de Δ -module). Dacă $\omega : I \rightarrow X$, atunci în baza proprietății de drum unic ridicat a unei proiecții de acoperire, pentru fiecare $\tilde{x} \in p^{-1}(\omega(0))$ există un drum $\tilde{\omega}_{\tilde{x}} : I \rightarrow \tilde{X}$ cu $\tilde{\omega}_{\tilde{x}}(0) = \tilde{x}$ și $p\tilde{\omega}_{\tilde{x}} = \omega$. Putem considera omomorfismul $\tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}) : \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}(0)) \rightarrow \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}(1))$ și cum $\Gamma(\omega(0)) = \prod_{\tilde{x} \in p^{-1}(\omega(0))} \tilde{\Gamma}(\tilde{x}) = \prod_{\tilde{x} \in p^{-1}(\omega(0))} \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}(0))$, $\Gamma(\omega(1)) = \prod_{\tilde{x} \in p^{-1}(\omega(1))} \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}(1))$, (dacă $\tilde{x} \in p^{-1}(\omega(1))$ există $\tilde{\omega}_{\tilde{x}}^{-1}$ ridicare în $\tilde{x}$ a lui ω^{-1} și avem $\tilde{\omega}_{\tilde{x}}^{-1} = \tilde{\omega}_{\tilde{x}}^{-1}$), putem defini $\Gamma(\omega) = \prod_{\tilde{x} \in p^{-1}(\omega(0))} \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}) : \prod \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}(0)) \rightarrow \prod \tilde{\Gamma}(\tilde{\omega}_{\tilde{x}}(1))$. În urma unor calcule simple se obține că (X, Γ) , cu $\Gamma(x)$ și $\Gamma(\omega)$ definite mai sus, este un obiect în $\mathcal{C}$.

Să arătăm acum că $p : \tilde{X} \rightarrow X$ definește un morfism în categoria $\mathcal{C}$, $(p, p_{\tilde{x}, \tilde{x} \in \tilde{X}}) : (\tilde{X}, \tilde{\Gamma}) \rightarrow (X, \Gamma)$, unde am notat prin $p(\tilde{\Gamma})$ sistemul local de module Γ , de mai sus. Anume, definim pentru fiecare $\tilde{x} \in \tilde{X}$ omomorfismul $p_{\tilde{x}}$ încît $\pi_{\tilde{x}} p_{\tilde{x}} = \text{Id}$ și $\pi_{\tilde{y}} p_{\tilde{x}} = 0$, dacă $\tilde{x} \neq \tilde{y}$, unde $\pi_{\tilde{y}} : \prod_{\tilde{y} \in p^{-1}(p(\tilde{x}))} \tilde{\Gamma}(\tilde{y}) \rightarrow \tilde{\Gamma}(\tilde{y})$ este proiecția canonică.

Obținem :

Propoziția 3. Dacă $p: \tilde{X} \rightarrow X$ este o proiecție de acoperire și $\tilde{\Gamma}$ este un sistem local de Δ -module pe spațiul $\tilde{X}$, atunci există un sistem local de Δ -module, $p(\tilde{\Gamma})$, pe X și pentru fiecare $\tilde{x} \in \tilde{X}$ un omomorfism $p_{\tilde{x}}: \tilde{\Gamma}(\tilde{x}) \rightarrow p(\tilde{\Gamma})(p(\tilde{x}))$ încît $(p, p_{\tilde{x}}, \tilde{x} \in \tilde{X})$ este un morfism în categoria $\mathcal{C}$.

Vom spune că $(p, p_{\tilde{x}}, \tilde{x} \in \tilde{X}): (\tilde{X}, \tilde{\Gamma}) \rightarrow (X, p(\tilde{\Gamma}))$ este o proiecție de acoperire în categoria $\mathcal{C}$.

Dacă $\tilde{\varphi}_0: \Delta \rightarrow \tilde{\Gamma}(\tilde{x}_0)$ este un omomorfism de Δ -module, atunci definim omomorfismul $\varphi_0: \Delta \rightarrow \Gamma(p(\tilde{x}_0)) = \prod_{\tilde{x} \in p^{-1}(p(\tilde{x}_0))} \tilde{\Gamma}(\tilde{x})$, încît $\pi_{\tilde{x}_0} \varphi_0 = \tilde{\varphi}_0$ și $\pi_{\tilde{x}} \varphi_0 = 0$, pentru $\tilde{x} \in p^{-1}(p(\tilde{x}_0))$ cu $\tilde{x} \neq \tilde{x}_0$. Morfismul $(p, p_{\tilde{x}})$ induce atunci omomorfismul $p_*: \pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_0) \rightarrow (X, x_0, p(\tilde{\Gamma}), p(\tilde{\varphi}_0))$. Mai general, dacă $(\tilde{X}, \tilde{\Gamma}, \tilde{\varphi}) \in \text{Ob } \mathcal{C}'$ atunci morfismul $\tilde{\varphi}: (\tilde{X}, \Delta) \rightarrow (\tilde{X}, \tilde{\Gamma})$ induce un morfism $p(\tilde{\varphi}): (X, \Delta) \rightarrow (X, p(\tilde{\Gamma}))$ astfel încît $(X, p(\tilde{\Gamma}), p(\tilde{\varphi})) \in \text{Ob } \mathcal{C}'$ și $(p, p_{\tilde{x}}, \tilde{x} \in \tilde{X})$ este un morfism în $\mathcal{C}'$. Rezultă un morfism de grupuri

$$p_*: \pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0}) \rightarrow \pi(X, p(\tilde{x}_0), p(\tilde{\Gamma}), (p\tilde{\varphi})_{p(\tilde{x}_0)}).$$

Lema 2. Dacă $\tilde{X}$ și X sînt spații liniar conexe și $(\tilde{\omega}, \tilde{\omega}_t), (\tilde{\omega}', \tilde{\omega}) \in \Omega(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0})$ încît $p\tilde{\omega} \simeq p\tilde{\omega}'$ în $\mathcal{C}$, atunci $\tilde{\omega} \simeq \tilde{\omega}'$ în $\mathcal{C}$.

Demonstrație. Fie $F: I \times I \rightarrow X$, $F_{(t,t')}: \Delta \rightarrow (p(\tilde{\Gamma}))(F(t, t'))$ care dau o omotopie în $\mathcal{C}$ a drumurilor $p\tilde{\omega}$ și $p\tilde{\omega}'$. Din proprietatea de ridicare a aplicației p , există o omotopie $F': I \times I \rightarrow X$ cu $pF' = F$ și $F': \omega \simeq \omega'$ în Top . Definim în plus, pentru fiecare $(t, t') \in I \times I$ omomorfismul $F_{(t,t')}: \Delta \rightarrow \tilde{\Gamma}(F'(t, t'))$, prin formula $F'_{(t,t')} = \pi_{F(t,t')} F_{(t,t')}$. Se constată ușor că F' și $F'_{(t,t')}$ dau o omotopie în $\mathcal{C}$ a drumurilor $\tilde{\omega}$ și $\tilde{\omega}'$.

Corolarul 4. Dacă $p: \tilde{X} \rightarrow X$ este o proiecție de acoperire, cu $\tilde{X}$ și X spații liniar conexe, și $(\tilde{X}, \tilde{\Gamma}, \tilde{\varphi}) \in \text{Ob } \mathcal{C}'$, atunci există un monomorfism $p_*: \pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0}) \rightarrow \pi(X, p(\tilde{x}_0), p(\tilde{\Gamma}), (p\tilde{\varphi})_{p(\tilde{x}_0)})$.

În mod analog cazului grupurilor fundamentale, se arată că indicele de acoperire coincide cu indicele subgrupului $p_*\pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0})$ în grupul $\pi(\tilde{X}, p(\tilde{x}_0), p(\tilde{\Gamma}), (p\tilde{\varphi})_{p(\tilde{x}_0)})$. Obținem :

Propoziția 4. Dacă $p: \tilde{X} \rightarrow X$ este o proiecție de acoperire regulată, $(\tilde{X}, \tilde{\Gamma}, \tilde{\varphi}) \in \text{Ob } \mathcal{C}'$, atunci există un izomorfism $\pi(X, p(\tilde{x}_0), p(\tilde{\Gamma}), (p\tilde{\varphi})_{p(\tilde{x}_0)})/p_*\pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0}) \simeq \pi(X, p(\tilde{x}_0))/p_*\pi(\tilde{X}, \tilde{x}_0)$, indus de monomorfismul φ_* din corolarul 1.

Corolarul 5. În condițiile propoziției 4, dacă X este spațiu simplu conex, atunci $\pi(X, p(\tilde{x}_0), (p\tilde{\Gamma}), (p\tilde{\varphi})_{p(\tilde{x}_0)}) \approx p_*\pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0})$.

Dacă $\tilde{X}$ este simplu conex atunci are loc izomorfismul

$$\pi(X, p(\tilde{x}_0), p(\tilde{\Gamma}), (p\tilde{\varphi})_{p(\tilde{x}_0)})/p_*\pi(\tilde{X}, \tilde{x}_0, \tilde{\Gamma}, \tilde{\varphi}_{\tilde{x}_0}) \simeq \pi(X, p(\tilde{x}_0)).$$

Ca o aplicație, să considerăm proiecția de acoperire $\exp: R \rightarrow S^1$ definită prin $\exp(t) = e^{2\pi i t}$. Dacă $(R, \Gamma, \varphi) \in \text{Ob } \mathcal{C}'$ atunci, din propoziția 4 și corolarul 3 obținem

Corolarul 6. *Oricare ar fi un sistem local de Δ -module pe R , $\varphi = (1_R, \varphi): (R, \Delta) \rightarrow (R, \Gamma)$ un morfism în $\mathcal{C}$, are loc izomorfismul $\pi(S^1, \exp(\Gamma), \exp(\varphi)) \simeq \mathbb{Z}$.*

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SUR LE GROUPE DES CLASSES DES CHEMINS DANS UN ESPACE TOPOLOGIQUE AVEC SYSTÈME LOCAL DE Δ -MODULES

(Résumé)

On considère la catégorie $\mathcal{C}$ des espaces topologiques avec des systèmes locaux de Δ -modules. On définit les chemins dans un objet $(X, \Gamma) \in \text{Ob } \mathcal{C}$ comme étant des morphismes de (I, Δ) en (X, Γ) . On considère les chemins φ_0 fermés dans $x_0 \in X$ et on démontre que les classes d'homotopie dans $\mathcal{C}$ de ces chemins forment un groupe $\pi(X, x_0, \Gamma, \varphi_0)$. On établit certaines propriétés de ce groupe. En considérant des projections de relèvement dans la catégorie $\mathcal{C}$ on établit la liaison avec les groupes attachés à l'espace de base et l'espace total.

ASUPRA UNEI CLASE DE ALGEBRE CHIMICE DE ORDINUL DOI

DE
ION ENESCU și IOLANDA ENESCU

1. În [1] se arată că algebra A asociată, în sensul din [2], cineticii de reacție pentru reacțiile inverse de ordin unu cu reacție directă de ordin doi, este reprezentată, într-o bază u_1, u_2 , prin tabela

$$(1) \quad \begin{array}{c|cc} & u_1 & u_2 \\ \hline u_1 & -ku_1 + \frac{k}{2}u_2 & \frac{1}{2}u_1 - \frac{1}{4}u_2 \\ u_2 & \frac{1}{2}u_1 - \frac{1}{4}u_2 & 2u_1 - u_2 \end{array},$$

unde k este un parametru real strict pozitiv.

În această Notă ne propunem studiul structurii acestei clase de algebre, după toate valorile reale ale parametrului k .

2. Schimbînd baza în A după

$$v_1 = u_1 + \frac{u_2}{2}, \quad v_2 = u_1 - \frac{u_2}{2}$$

se obține tabela

$$(1') \quad \begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & (1-k)v_2 & -\left(k + \frac{1}{2}\right)v_2 \\ v_2 & -\left(k + \frac{1}{2}\right)v_2 & -kv_2 \end{array}$$



care pune în evidență faptul că ansamblul elementelor de forma $\{\alpha v_2\}$, cu α real arbitrar, formează un ideal propriu.

Cărestarea asociativității produsului, arată că algebra A are această proprietate numai pentru $k = -\frac{1}{8}$.

Fie $x = av_1 + bv_2$ un element arbitrar din A . Ecuația $x^2 = 0$ conduce, excluzând cazul $x = 0$, la soluțiile

$$x_1 = b \left[\frac{2k+1+\sqrt{8k+1}}{2(1-k)} v_1 + v_2 \right], \quad x_2 = b \left[\frac{2k+1-\sqrt{8k+1}}{2(1-k)} v_1 + v_2 \right],$$

cu b scalar arbitrar și $k \neq 1$.

Observăm că pentru $k = -\frac{1}{8}$, $x_1 = x_2 = b \left(\frac{1}{3} v_1 + v_2 \right)$.

În cazul $k = 1$, aceeași ecuație are soluțiile

$$x_1 = v_1 \quad \text{și} \quad x_2 = b \left(-\frac{1}{3} v_1 + v_2 \right),$$

ceea ce stabilește

Teorema 1. Algebra (1) are, pentru orice k real și diferit de $-\frac{1}{8}$, două elemente nilpotente proprii, independente, $\varepsilon_1, \varepsilon_2$.

Nilpotenții $\varepsilon_1, \varepsilon_2$ sînt deei

$$(2) \quad \begin{cases} \varepsilon_1 = \frac{2k+1+\sqrt{8k+1}}{2(1-k)} v_1 + v_2, \\ \varepsilon_2 = \frac{2k+1-\sqrt{8k+1}}{2(1-k)} v_1 + v_2, \end{cases} \quad k \neq 1; k \neq -\frac{1}{8}$$

și

$$(2') \quad \begin{cases} \varepsilon_1 = v_1, \\ \varepsilon_2 = -\frac{1}{3} v_1 + v_2, \end{cases} \quad k = 1.$$

Căutînd acum elementele idempotente din A , se constată că ecuația $x^2 = x$, are, excluzînd cazul $x = 0$, numai soluția

$$(3) \quad E = -\frac{1}{k} v_2, \quad k \neq 0.$$

Observăm că în cazul $k = 0$, algebra A nu are nici un idempotent propriu, ceea ce dovedește

Teorema 2. Algebra A corespunzătoare valorii $k = 0$, nu este izomorfă cu nici o alta, obținută pentru valori diferite de zero ale parametrului.

3. Să determinăm în continuare, toate tipurile neizomorfe de algebre A , pentru $k \neq 0$.

Pentru aceasta să considerăm baza formată cu elementele nilpotente date de (2). Se obține, în acest caz, tabela de înmulțire

$$(4) \quad \begin{array}{c|cc} & \varepsilon_1 & \varepsilon_2 \\ \hline \varepsilon_1 & 0 & \frac{1-2k}{4(1-k)\sqrt{8k+1}} \left[(\sqrt{8k+1}-2k-1)\varepsilon_1 + (\sqrt{8k+1}+2k+1)\varepsilon_2 \right] \\ \varepsilon_2 & \frac{1-2k}{4(1-k)\sqrt{8k+1}} \left[(\sqrt{8k+1}-2k-1)\varepsilon_1 + (\sqrt{8k+1}+2k+1)\varepsilon_2 \right] & 0 \end{array}$$

unde $k \neq 1$, $k \neq -\frac{1}{8}$.

Fie A_{k_1} și A_{k_2} două algebre A , corespunzătoare la respectiv două valori diferite k_1 și k_2 ale parametrului, ($k_1, k_2 \neq 1, \neq -\frac{1}{8}$), a căror tabelă de înmulțire este dată de (4) unde, parametrul k este egal cu respectiv k_1 și k_2 .

Notînd cu $\varepsilon_1^{(1)}, \varepsilon_2^{(1)}$ și respectiv $\varepsilon_1^{(2)}, \varepsilon_2^{(2)}$ bazele în A_{k_1} și A_{k_2} , pentru ca aceste algebre să fie izomorfe, este necesar ca

$$(5) \quad \varphi[\varepsilon_1^{(1)} \varepsilon_2^{(1)}] = \varepsilon_1^{(2)} \varepsilon_2^{(2)},$$

unde φ este un operator care realizează un izomorfism între cele două algebre, ales astfel, încît $\varphi[\varepsilon_1^{(1)}] = \varepsilon_1^{(2)}$ și $\varphi[\varepsilon_2^{(1)}] = \varepsilon_2^{(2)}$.

Observăm că această alegere a corespondenței între elementele bazelor celor două algebre, nu restrînge generalitatea rezultatului, întrucît, pe de o parte, elementele ε_1 și ε_2 sînt nilpotente și, pe de altă parte $\varepsilon_1 \varepsilon_2 = \varepsilon_2 \varepsilon_1$.

Condiția (5) conduce la relațiile

$$(6) \quad \begin{cases} \frac{1-2k_1}{4(1-k_1)\sqrt{8k_1+1}} (\sqrt{8k_1+1}-2k_1-1) = \frac{1-2k_2}{4(1-k_2)\sqrt{8k_2+1}} (\sqrt{8k_2+1}-2k_2-1), \\ \frac{1-2k_1}{4(1-k_1)\sqrt{8k_1+1}} (\sqrt{8k_1+1}+2k_1+1) = \frac{1-2k_2}{4(1-k_2)\sqrt{8k_2+1}} (\sqrt{8k_2+1}+2k_2+1), \end{cases}$$

care se pot scrie sub forma

$$(6') \quad \begin{cases} \frac{1-2k_1}{1-k_1} - \frac{(1-2k_1)(1+2k_1)}{(1-k_1)\sqrt{8k_1+1}} = \frac{1-2k_2}{1-k_2} - \frac{(1-2k_2)(1+2k_2)}{(1-k_2)\sqrt{8k_2+1}}, \\ \frac{1-2k_1}{1-k_1} + \frac{(1-2k_1)(1+2k_1)}{(1-k_1)\sqrt{8k_1+1}} = \frac{1-2k_2}{1-k_2} + \frac{(1-2k_2)(1+2k_2)}{(1-k_2)\sqrt{8k_2+1}}. \end{cases}$$

Adunarea relațiilor (6') conduce la

$$\frac{1 - 2k_1}{1 - 2k_2} = \frac{1 - k_1}{1 - k_2},$$

care nu poate avea loc decât pentru $k_1 = k_2$, ceea ce demonstrează

Teorema 3. *Toate algebrele A corespunzătoare valorilor k ($k \neq 1$; $\neq -\frac{1}{8}$) sînt neizomorfe.*

Rămîn în discuție cazurile $k = -\frac{1}{8}$ și $k = 1$.

Pentru $k = -\frac{1}{8}$, algebra A este asociativă, aceasta fiind singura algebră (1) care are această proprietate și, prin urmare, este neizomorfă cu oricare alta.

Notînd cu respectiv A_1 și A_k algebra A corespunzătoare lui $k = 1$ și unei valori arbitrare $k \neq 1$, condiția (5) și corespondența dintre elementul idempotent (3) pentru A_1 și A_k , conduc la valori diferite pentru un același k , ceea ce arată că A_1 și A_k ($k \neq 1$), nu pot fi izomorfe. Rezultatele din acest paragraf, împreună cu teorema 2, stabilesc

Teorema 4. *Toate algebrele (1), obținute pentru orice k real, sînt neizomorfe.*

4. Pentru cazul asociativ ($k = -\frac{1}{8}$), cei doi nilpotenți coincid cu $N = \frac{1}{3}v_1 + v_2$ iar elementul idempotent (3) are forma $E = 8v_2$. În baza N, E , algebra are tabela

	E	N
E	E	0
N	0	0

și, se vede că ea are structura de sumă directă dintre algebra numerelor reale și algebra nulă de ordinul unu.

5. Structura algebrei pentru cazul $k = -\frac{1}{8}$ arată și modul în care

se pot construi toate algebrele A pentru $k \neq 0$. Acestea sînt obținute ca sumă dintre algebra numerelor reale și algebra nulă de ordinul unu, produsul dintre elementele acestora fiind dat arbitrar, ceea ce explică pe deplin, absența oricăror tipuri izomorfe de algebre din familia (1).

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SUR UNE CLASSE D'ALGÈBRES CHIMIQUES D'ORDRE DEUX

(Résumé)

En considérant les réactions chimiques dont l'inverse est d'ordre un ayant réaction directe d'ordre deux, on détermine les types non-isomorphes d'algèbres associées (1), en mettant aussi en évidence leurs propriétés de structure.

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UPPER SEMI-CONTINUOUS FUNCTIONS AND QUASI-PROXIMITY CLASSES

BY

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1. Introduction

Upper semi-continuous (u. s. c.) functions occur frequently in the study of analysis and topology; e. g., as the limit of a decreasing sequence of real valued functions on a metric space. These mappings together with the quasiuniformly upper semi-continuous (q. u. u. s. c.) functions play a central role in the study of quasi-uniform spaces. In this note we obtain new characterizations of topological and quasi-uniform properties in terms of these mappings. Of course we could have used lower semi-continuous (l. s. c.) functions in these theorems. In fact, we found it convenient to use l. s. c. functions to characterize paracompactness in section 4.

Fenstad [5] showed that two uniform structures are p -equivalent if and only if they possess the same bounded real valued uniformly continuous functions. We show that the corresponding invariant for a quasi-proximity class of quasi-uniformities, $\Sigma(\delta)$, is the set of all bounded real valued functions which are q. u. u. s. c. with respect to $\mathcal{U}$, where $\mathcal{U}$ is any member of $\Sigma(\delta)$. To obtain this result, we extend the well known lemma of Smirnov [15] which states that any two far sets can be separated by a p -continuous function.

Given a quasi-uniformity $\mathcal{U}$ on X , Fletcher [6] constructs a totally bounded quasi-uniformity contained in $\mathcal{U}$. We show that this construction yields the totally bounded member of the quasi-proximity class determined by $\mathcal{U}$ [10].

In the final section we give an example of a quasi-uniformity which induces a proximity in the usual manner [10]. We consider the class $\Sigma(\delta)$ of all quasi-uniformities compatible with a proximity δ , and compare some of its properties to the proximity class of uniformities $\pi(\delta)$. In a pre-

vious work [10] the authors proved that for δ a quasi-proximity, $\Sigma(\delta)$ has a coarsest member. We give an example which shows that $\Sigma(\delta)$ need not have a finest member even if $\Sigma(\delta)$ contains a metrizable uniformity.

2. Preliminaries

If X is a nonempty set and $\mathcal{U}$ is a filter on $X \times X$, then $\mathcal{U}$ is a quasi-uniformity on X if and only if

- (i) For each $U \in \mathcal{U}$, $\Delta = \{(x, x) : x \in X\} \subset U$ and,
- (ii) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

A relation δ on $\mathcal{A}(X)$ is a quasi-proximity for X if and only if it satisfies

- (1) $A \delta \emptyset$ and $\emptyset \delta A$ for each A in $\mathcal{A}(X)$. ($A \delta B$ denotes the negation of $A \delta B$).
- (2) $C \delta (A \cup B)$ if and only if $C \delta A$ or $C \delta B$, and $(A \cup B) \delta C$ if and only if $A \delta C$ or $B \delta C$,
- (3) $\{x\} \delta \{x\}$ for each $x \in X$.
- (4) If $A \delta B$, then there exist C, D with $C \cap D = \emptyset$,

$$A \delta (X - C), \quad \text{and} \quad (X - D) \delta B.$$

For $\varepsilon > 0$ and f a real valued function on X , let $U(\varepsilon, f) = \{(x, y) : (fy) - f(x) < \varepsilon\}$. If $\mathcal{U}$ is a quasi-uniformity on X , then f is q. u. u. s. c. with respect to $\mathcal{U}$ if and only if $U(\varepsilon, f) \in \mathcal{U}$ for every $\varepsilon > 0$.

A quasi-uniformity $\mathcal{U}$ on a set X is totally bounded if and only if for every U in $\mathcal{U}$, there exists a finite cover $\{A_i\}$ of X with $A_i \times A_i \subset U$ for each i . $\mathcal{U}$ is precompact if for every U in $\mathcal{U}$, there exists a finite subset F of X with $U(F) = X$. $\mathcal{U}$ is bounded if for every V in $\mathcal{U}$ there exists a positive integer n and a finite subset F of X with $V^n(F) = X$.

To every quasi-proximity space (X, δ) there corresponds a unique totally bounded quasi-uniform space $(X, \mathcal{U}_\delta)$. A subbase for $\mathcal{U}_\delta$ consists of all sets of the form $T(A, B) = X \times X - A \times B$, where $A \delta B$. A base for $\mathcal{U}_\delta$ consists of all sets of the form

$$U\{B_i \times A_i : 1 \leq i \leq n\},$$

where $\{B_i : 1 \leq i \leq n\}$ is a finite cover of X and $B_i \delta (X - A_i)$, for every i , $1 \leq i \leq n$. Let $\mathcal{V}$ be the quasi-uniformity on the set of real numbers, R , with base $\{V_\varepsilon : \varepsilon > 0\}$, where $V_\varepsilon = \{(x, y) : y - x < \varepsilon\}$. Let β be the quasi-proximity induced on R by $\mathcal{V}$. We will also use the symbols $\mathcal{V}$ and β for their restrictions to subsets of R .

Let $(X, \mathcal{U})$ be a quasi-uniform space, and let f be a real valued function on X . Clearly f is q. u. u. s. c. with respect to $\mathcal{U}$ if and only if f is $\mathcal{U} - \mathcal{V}$ quasi-uniformly continuous. Note also that f is p -continuous from (X, δ) into (R, β) if and only if f is $\mathcal{U}_\delta - \mathcal{U}_\beta$ quasi-uniformly continuous.

3. Quasi-proximities

The following is a generalization of Smirnov's Lemma to quasi-proximity spaces. The proof of it is based on [6, Th. 6] and [4, Th. 52].

3.1. Lemma. *Let (X, δ) be a quasi-proximity space. If $A\delta-B$, there exists a function g from X into $[0,1]$ such that g is q. u. u. s. c. with respect to $\mathcal{U}_\delta$, and $g(A) = \{0\}$, $g(\bar{B}) = \{1\}$.*

Proof. Let $A\delta-B$. Then $A\delta-\bar{B}$, and there exists $V_0 \in \mathcal{U}_\delta$ with $V_0(A) \subset X - \bar{B}$. Let (V_k) be a sequence of elements in $\mathcal{U}_\delta$ satisfying $V_{k+1}^2 \subset V_k$, $(k = 0, 1, 2, \dots)$. If $0 \leq t \leq 1$, and $t = \sum_{k=1}^{\infty} d_k(t) 2^{-k}$, where $d_k(t) = 0$ or 1 , let $D_t = \{k : d_k(t) = 1\}$ and $T = \{t : 0 \leq t \leq 1 \text{ and } D_t \text{ is finite}\}$. Let $U_0 = \Delta$, and for any $t \in T$, let $U_t = V_{k_n} V_{k_{n-1}} \dots V_{k_1}$, where $\{k_1, \dots, k_n\} = D_t$ and $k_1 < k_2 < \dots < k_n$. In [4, p. 124] it is proved that if $t = p 2^{-m}$ and $t' = (p+1) 2^{-m}$ with t and t' in T , then $V_m U_t \subset U_{t'}$. Note also that if $a < b$, $U_a \subset U_b$.

Define g by $g(x) = 0$ if $x \in A$ and $g(x) = \sup \{t \in T : x \notin U_t(A)\}$ if $x \notin A$. Let $\varepsilon > 0$ and w be a positive integer such that $2^{-w+1} < \varepsilon$. Let $(y, z) \in V_w$ and suppose $g(z) - g(y) \geq \varepsilon$. Then $z \notin A$. Since $g(z) \leq 1$, $g(y) < (2^w - 1) 2^{-w}$. Let p be the least positive integer such that $g(y) < p 2^{-w}$ or $g(z) < p 2^{-w}$. It follows that $p \leq 2^w - 1$, therefore $(p+1) 2^{-w} \in T$. Suppose $g(y) < p 2^{-w}$. There exists $t \in T$ such that $y \in U_t(A)$ and $t \leq p 2^{-w}$. Therefore $y \in U_t(A) \subset U_{p 2^{-w}}(A)$ so $V_w(y) \subset V_w[U_{p 2^{-w}}(A)]$. Since $V_w U_{p 2^{-w}} \subset U_{(p+1) 2^{-w}}$, we have $V_w(y) \subset U_{(p+1) 2^{-w}}(A)$. Now $g(z) \leq (p+1) 2^{-w}$ since $z \in V_w(y)$. But $g(y) \geq (p-1) 2^{-w}$ and $g(z) \leq (p+1) 2^{-w}$ implies $g(z) - g(y) \leq 2^{-w+1} < \varepsilon$, and this contradicts our first assumption. Therefore $g(z) < p 2^{-w}$. We have $p 2^{-w} > g(z) \geq g(y) + \varepsilon \geq (p-1) 2^{-w} + \varepsilon$ this contradicts $2^{-w+1} < \varepsilon$. It follows that $\{(y, z) : g(z) - g(y) < \varepsilon\} \in \mathcal{U}_\delta$; hence g is q. u. u. s. c. with respect to $\mathcal{U}_\delta$. Since $g(A) = \{0\}$, and $g(\bar{B}) = \{1\}$, the proof is complete.

Let $I = [0, 1]$. Note that the function g constructed in the proof of Lemma 3.1 is p -continuous from (X, δ) into (I, β) .

If $(X, \mathcal{U})$ is a quasi-uniform space, $Q(\mathcal{U})$ ($Q_B(\mathcal{U})$) will denote the set of all (bounded) q. u. u. s. c. functions from $(X, \mathcal{U})$. $P(\delta)$ ($P_B(\delta)$) will denote the set of all (bounded) p -continuous functions from (X, δ) into (R, β) . In [6] Fletcher shows that set of all $U(\varepsilon, f)$ such that $\varepsilon > 0$ and f is in $Q_B(\mathcal{U})$ is a subbase for a totally bounded quasi-uniformity $\mathcal{U}'$ contained in $\mathcal{U}$.

3.2. Proposition. *If (X, U) is bounded, then every f in $Q(\mathcal{U})$ is bounded above.*

Recall that a uniform space is bounded if and only if every real valued uniformly continuous function is bounded [2]. In view of this it seems natural to conjecture the converse of Proposition 3.2. The following example shows that this conjecture is false.

It is evident that a quasi-uniformity with a base of transitive entourages (i. e. a transitive quasi-uniformity) is precompact if and only if it is bounded. In [9] it is proved that a topological space is compact if and only if the finest compatible transitive quasi-uniformity, $\mathcal{T}\mathcal{S}$, is precompact.

3.3. Example. Let Ω denote the first uncountable ordinal and let X be the set of all ordinals less than Ω . Let $\mathcal{S} = \{[0, x) : x \in X - \{0\}\}$ together with $\emptyset$ and X . $(X, \mathcal{S})$ is countably compact but not compact. Since it is countably compact every u. s. c. function is bounded above [3, lemma 3.2.]. Let f be any u. s. c. function on X . Since 0 is in every nonempty open set, it follows that for any $x \in X$, $f(x) \geq f(0)$. Clearly $Q(\mathcal{T}\mathcal{S}) = Q_B(\mathcal{T}\mathcal{S})$. Since X is not compact, however, $\mathcal{T}\mathcal{S}$ is not bounded.

3.4. Theorem. Let (X, δ) be a quasi-proximity space. Let $\mathcal{U}$ be a quasi-uniformity compatible with δ . Then $Q(\mathcal{U}_\delta) = Q_B(\mathcal{U}) = P_B(\delta)$.

Proof. Let $f \in Q(\mathcal{U}_\delta)$. Since $\mathcal{U}_\delta$ is totally bounded it is bounded, hence f is bounded above. Suppose f is unbounded below. Then there exists a sequence (x_i) in X such that $f(x_i) - 1 > f(x_{i+1})$. $U(1/2, f)$ is totally bounded and hence there exists a finite cover $\{A_i : 1 \leq i \leq n\}$ of X with $\bigcup \{A_i \times A_i : 1 \leq i \leq n\} \subset U(1/2, f)$. There exist positive integers i, j and k , ($j > k$), ($1 \leq i \leq n$) such that x_j and x_k are in A_i , and $f(x_j) < f(x_k) - 1$. Since $(x_j, x_k) \in A_i \times A_i$, $(x_j, x_k) \in U(1/2, f)$ and hence $f(x_i) - f(x_j) < 1/2$. This contradicts the previous inequality. Hence $Q(\mathcal{U}_\delta) = Q_B(\mathcal{U}_\delta)$. Let $f \in Q(\mathcal{U}_\delta)$. Let A and B be subsets of R with $A\beta(X - B)$. There exists $\varepsilon > 0$ such that $V_\varepsilon(A) \subset B$. From $U(\varepsilon, f)(f^{-1}(A)) \subset f^{-1}(B)$ it follows that $f \in P_L(\delta)$.

Let $f \in P_L(\delta)$. Then $f(X)$ is contained in an interval $[a, b]$. The restriction of $\mathcal{V}$ to $[a, b]$ is totally bounded, and since total boundedness is hereditary, the restriction of $\mathcal{V}$ to $f(X)$ is totally bounded. It follows from earlier remarks that f is $\mathcal{U}_\delta$ -quasi-uniformly continuous, that is $f \in Q(\mathcal{U}_\delta)$. Hence $Q(\mathcal{U}_\delta) = P_L(\delta)$. Since every quasi-uniformly continuous function is p -continuous, we have $Q_B(\mathcal{U}) \subset P_B(\delta)$. From $\mathcal{U}_\delta \subset \mathcal{U}$, it follows that $Q(\mathcal{U}_\delta) \subset Q_B(\mathcal{U})$. Hence $Q(\mathcal{U}_\delta) = Q_B(\mathcal{U})$.

3.5. Corollary. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are quasi-uniformities on X , then $\mathcal{U}_1$ and $\mathcal{U}_2$ are in the same quasi-proximity class if and only if $Q_B(\mathcal{U}_1) = Q_B(\mathcal{U}_2)$.

3.6. Corollary. If (X, δ) is a quasi-proximity space, and $\mathcal{U}$ is any compatible quasi-uniformity, then $\{\mathcal{U}(\varepsilon, f) : \varepsilon > 0 \text{ and } f \in Q_B(\mathcal{U})\}$ is a subbase for $\mathcal{U}_\delta$.

Proof. Let $\mathcal{U}'$ be the quasi-uniformity on X with subbase as in hypothesis. In [6] Fletcher shows that $\mathcal{U}'$ is totally bounded and $\mathcal{U}' \subset \mathcal{U}$. Since $\mathcal{U}_\delta$ is the finest totally bounded quasi-uniformity contained in $\mathcal{U}$, it follows that $\mathcal{U}' \subset \mathcal{U}_\delta$. Hence $Q_B(\mathcal{U}') \subset Q_B(\mathcal{U}_\delta)$. If $f \in Q_B(\mathcal{U}_\delta)$, then $f \in Q_B(\mathcal{U})$ and by definition of $\mathcal{U}'$ we have $f \in Q_B(\mathcal{U}')$. By Corollary 3.5 $\mathcal{U}'$ is compatible with δ and since there is a unique totally bounded quasi-uniformity in $\Sigma(\delta)$, we have $\mathcal{U}' = \mathcal{U}_\delta$.

3.7. Corollary. *If $\mathcal{U}$ is totally bounded then every $q. u. u. s. c.$ function from $(X, \mathcal{U})$ is bounded.*

Example 3.3 also shows that the converse of Corollary 3.7 is false. Since if $\mathcal{F}\mathcal{F}$ were totally bounded it would necessarily be bounded.

4. Paracompact Spaces

One of the useful properties of paracompact spaces is the existence of families of functions called partitions of unity. Partitions of unity have been used to prove Shirota's Theorem: A Tihonov space in which every closed discrete subset has nonmeasurable cardinal, is realcompact if and only if it admits a compatible complete uniformity. In [9] it is proved that a normal countably paracompact space is realcompact if and only if the quasi-uniformity generated by the lower semi-continuous functions is complete. In light of this the following theorem may facilitate further investigations of realcompactness.

A family $\{f_\lambda : \lambda \in \Lambda\}$ of l. s. c. (continuous) functions defined on a space X with values in $[0, 1]$ is called an l. s. c. (continuous) partition of unity on X if $\sum_{\lambda \in \Lambda} f_\lambda(x) = 1$ for each x in X . The equality means that for each x_0 in X only countably many f_λ assume values different from zero at x_0 and that the series $\sum_{i=1}^{\infty} f_{\lambda_i}(x_0)$, where $\{\lambda_1, \lambda_2, \dots\} = \{\lambda : f_\lambda(x_0) \neq 0\}$, is (absolutely) convergent and its sum is 1.

Let $C = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X an l. s. c. (continuous) partition of unity $\{f_\lambda : \lambda \in \Lambda\}$ is subordinate to C if the cover $\{f_\lambda^{-1}((0, 1]) : \lambda \in \Lambda\}$ refines C .

A proof of the following lemma for continuous partitions of unity may be found in [12].

4.1. Lemma. *Let $\{f_\lambda : \lambda \in \Lambda\}$ be an l. s. c. partition of unity on X , $x \in X$ and $\delta > 0$. Then there exists a neighborhood of x on which only a finite number of f_λ s take on a value greater than or equal δ .*

Proof. Let Λ' be a finite subset of Λ such that $\sum_{\lambda \in \Lambda'} f_\lambda(x) > 1 - \frac{\delta}{2}$, and set $n = |\Lambda'|$. Set $G = \bigcap_{\lambda \in \Lambda'} f_\lambda^{-1}\left(f_\lambda(x) - \frac{\delta}{2n}, 1\right)$, then for each y in G and for each λ in Λ' , $f_\lambda(y) > f_\lambda(x) - \frac{\delta}{2n}$. Then for each y in G , $\sum_{\lambda \in \Lambda'} f_\lambda(y) > \sum_{\lambda \in \Lambda'} \left(f_\lambda(x) - \frac{\delta}{2n}\right) = \sum_{\lambda \in \Lambda'} f_\lambda(x) - \frac{\delta}{2} > \left(1 - \frac{\delta}{2}\right) - \frac{\delta}{2} = 1 - \delta$. Hence for each λ in $\Lambda - \Lambda'$ and for each y in G , $f_\lambda(y) < \delta$.

4.2. Theorem. *Let X be a regular space. X is paracompact if and only if each open cover of X has an l. s. c. partition of unity subordinate to it.*

Proof. If X is paracompact, then each open cover has a continuous partition of unity subordinate to it [13]. Conversely let $C = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover and $\{f : \lambda \in \Lambda\}$ be an l. s. c. partition of unity subordinate to it. For each n in N (the set of positive integers) and for each λ in Λ , set $F(\lambda, n) = \{x \in X : f_\lambda(x) > 1/n\}$. Since $\{f_\lambda : \lambda \in \Lambda\}$ is an l. s. c. partition of unity, $\{F(\lambda, n) : \lambda \in \Lambda \text{ and } n \in N\}$ refines C . Moreover, by the above lemma, $\{F(\lambda, n) : \lambda \in \Lambda\}$ is an open locally finite collection for each n in N . Hence C has an open σ -locally finite refinement and X is paracompact [13].

5. Proximity Classes of Quasi-uniformities

5.1. Example. A quasi-uniformity $\mathcal{U}$ which is not a uniformity but is compatible with a proximity. Let X be the open unit interval and for each positive integer n , set $U_n = \Delta \cup \{(x, y) : 0 \leq x \leq y < 1/n\}$. The collection of all U_n for n a positive integer is a base for a quasi-uniformity $\mathcal{U}$ which is compatible with the discrete topology on X . Let δ denote the quasi-proximity induced on X by $\mathcal{U}$. Observe that $A \delta B$ if and only if $\text{glb } A = \text{glb } B = 0$ or $A \cap B \neq \emptyset$.

It is clear that the relation δ given above is symmetric, and hence δ is a proximity. It is also clear that $\mathcal{U}$ is not a uniformity.

5.2. Proposition. *A quasi-proximity δ is a proximity if and only if, for every $\mathcal{U} \in \Sigma(\delta)$, we have $\mathcal{U}^{-1} \in \Sigma(\delta)$.*

5.3. Corollary. *If δ is a proximity, and $\Sigma(\delta)$ has a finest member $\mathcal{U}$, then $\mathcal{U}$ is a uniformity.*

In view of the above corollary, the example given in [1] shows that $\Sigma(\delta)$ need not have a finest member. The following example shows that the metrizable member of a proximity class need not be the finest quasi-uniformity compatible with a given proximity.

5.4. Example. Let X be the open unit interval and for n a positive integer, let

$$W_n = \Delta \cup \{(x, y) : 0 < x < 1/n \text{ and } 0 < y < 1/n\}.$$

The set of all W_n for n a positive integer is a base for a uniformity $\mathcal{W}$ on X which is compatible with the proximity δ of example 5.1. Note that $\mathcal{W}$ is metrizable and hence is the finest uniformity in its proximity class [1]. However $\mathcal{W}$ is strictly coarser than the quasi-uniformity $\mathcal{U}$ of example 5.1. Note also that $\mathcal{U}$ is quasi-metrizable [14]. By Corollary 5.3 $\Sigma(\delta)$ contains no finest quasi-uniformity.

In the proof of the following theorem we use Theorem I [11]. We will use the following terminology from [11]. Let (X, δ) be a proximity

space. A subset P of $X \times X$ is compressed if and only if P is infinite and $A \delta B$ whenever $A \times B \cap P$ is infinite. Let U be a reflexive relation on X , U is admissible if and only if $U \cap P \neq \emptyset$ for every compressed set P . In the following we also use the fact that an infinite set P is compressed if and only if every entourage of the form $T(A, B)$ where $A \delta B$ contains all but finitely many points of P [11, Th. III].

5.5. Theorem. *Let (X, δ) be a proximity space and let $\mathcal{U}$ be a metrizable uniformity on X compatible with δ . Every quasi-metrizable quasi-uniformity compatible with δ is finer than $\mathcal{U}$.*

Proof. Since δ is metrizable [11], there exists a sequence (U_n) of admissible symmetric relations such that $U_{n+1} \subset U_n$, and $A \delta B$ if and only if

$$(A \times B) \cap U_n \neq \emptyset$$

for every n . In the proof of Theorem I [11], we see that the set of all U_n is a base for the metrizable uniformity compatible with δ . Let $\mathcal{V}$ be any quasi-metrizable quasi-uniformity compatible with δ , and assume $\mathcal{U} \not\subset \mathcal{V}$. Let $U \equiv \mathcal{U} - \mathcal{V}$. There exists an integer N such that if $n \geq N$, then $U_n \subset U$. Hence for these U_n we have $U_n \not\subset \mathcal{V}$. Let d be a quasi-metric which yields $\mathcal{V}$. Put

$$W_k = \{(x, y) : d(x, y) < 1/k\},$$

for k a positive integer. Since $U_N \not\subset \mathcal{V}$, we have $W_i \not\subset U_N$ for every i . Let P be a set formed by choosing one point $(x_i, y_i) \in W_i - U_N$ for each i . Clearly P is infinite. Suppose $A \delta B$, then $T(A, B) \equiv \mathcal{U}_\delta \subset \mathcal{V}$. There exists an integer M such that if $n \geq M$, we have $W_n \subset T(A, B)$. Therefore $T(A, B)$ contains almost all of P , and P is compressed. U_N is admissible and hence $U_N \cap P \neq \emptyset$ but by definition of P , $U_N \cap P = \emptyset$. This contradiction establishes the theorem.

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FUNCȚII SUPRA SEMICONTINUE ȘI CLASE CUASI-PROXIME

(Rezumat)

Se extinde la spații cuasi-proxime lema lui Smirnov care afirmă că orice pereche de șiruri îndepărtate pot fi separate printr-o δ -aplicație. Cu ajutorul său se stabilesc condițiile necesare și suficiente pentru ca două cuasi-uniformități să aparțină aceleiași clase cuasi-proxime. Se obține o nouă caracterizare a paracompactității, cu ajutorul funcțiilor sub semicontinue și se studiază structura de ordine a unei clase cuasi-proxime de cuasi-uniformități.

SUR DES CERTAINES STRUCTURES TOPOLOGIQUES DANS UN ANNEAU DE SÉRIES FORMELLES (II)

PAR

DORIN CIUMAȘU

1. Introduction

Dans la première partie de cet étude [1] on a exposé un certain nombre de résultats algébriques relatifs à un anneau de séries formelles et on a démontré que toute série formelle vectorielle peut être identifiée avec une série formelle à coefficients vectoriels. Après avoir étudié les topologies ordonnées définies, on a introduit les applications de substitution φ_r , et de composition régulière ψ_u et enfin on a étudié l'image de la composition régulière de deux séries par l'application de dérivation partielle D' .

Dans cette Note on étudie les applications φ_r et ψ_u , dans la topologie forte et dans la topologie faible. On donne ensuite une nouvelle démonstration pour un résultat relatif à la résolution des équations dans un anneau de séries formelles [2, § 5, no 9] et on étudie la topologie faible dans $A[[X_i]]_{i \in I}$, A étant un anneau topologique valué nondiscret. Enfin, on démontre que dans ce cas $A[[X_i]]_{i \in I}$ est un espace pseudométrique (proposition 9) et l'application D' est bornée (proposition 10).

2. Les applications de substitutions et de compositions dans la topologie forte

La continuité des applications de substitution et de composition quand les anneaux de séries formelles respectives sont munis avec les topologies fortes résulte aussitôt de la proposition suivante :

Proposition 4. Soient $u(X) \in A[[X_i]]_{i \in I}$ et $V(Y) = (v_i(Y))_{i \in I}$, $W(Y) = (w_i(Y))_{i \in I}$, avec $v_i(Y)$, $w_i(Y) \in A[[Y_j]]_{j \in J}$, $i \in I$ et $\omega_{J_1}(V) \geq 1$, $\omega_{J_1}(W) \geq 1$, $J_1 \subset J$. Alors on a :

$$1^\circ \omega_{J_1}(u(V(Y))) \geq \inf_{\substack{m \in J \\ |m| = \omega(u)}} \left\{ \sum_{i \in I} m_i \omega_{J_1}(v_i) \right\} \geq \omega(u).$$

$$2^\circ \text{ Si } \Omega(u(X)) = (\omega_i(u))_{i \in I}, \quad \Omega(v_i(Y)) = (\omega_k(v_i))_{k \in J}$$

alors

$$\Omega_{J_1}(u(V(Y))) = \left(\sum_{i \in I} \omega_i(u(X)) \omega_k(v_i(Y)) \right)_{k \in J_1}.$$

$$3^\circ \omega_{J_1}(u(V(Y)) - u(W(Y))) \geq \omega_{J_1}(V(Y) - W(Y)).$$

En effet la partie homogène de degré minimum de la série formelle $u = (u_m)_{m \in I} \in A[[X_i]]_{i \in I}$, est le polynôme homogène

$$\sum_{\substack{m \in I \\ |m| = \omega(u)}} u_m X^m.$$

Par suite la partie homogène de degré minimum de la série formelle $u(V(Y))$ est identique à celle de la série formelle,

$$\sum_{\substack{m \in I \\ |m| = \omega(u)}} u_m [V(Y)]^m.$$

Il résulte des propriétés de l'ordre partiel que,

$$\omega_{J_1}(u(V(Y))) = \omega_{J_1}([V(Y)]^m) = \sum_{i \in I} m_i \omega_{J_1}(v_i(Y)).$$

Et donc,

$$\begin{aligned} \omega_{J_1}(u(V(Y))) &= \omega_{J_1} \left(\sum_{\substack{m \in I \\ |m| = \omega(u)}} u_m [V(Y)]^m \right) \geq \\ &\geq \inf_{\substack{m \in I \\ |m| = \omega(u)}} \left\{ \sum_{i \in I} m_i \omega_{J_1}(v_i(Y)) \right\} \geq \inf_{\substack{m \in I \\ |u| = \omega(u)}} \left\{ \sum_{i \in I} m_i \right\} = \omega(u) \end{aligned}$$

ce qui démontre la relation 1° .

En tenant compte des propriétés de l'ordre strict, il s'ensuit de la même manière la relation 2° .

Parce que le terme de degré zéro de la série formelle $u(V(Y))$ est identique à celui de la série $u(W(Y))$, il résulte que la série formelle $u(V(Y)) - u(W(Y))$ est sans terme de degré zéro. Donc

$$\begin{aligned} \omega_{J_1}(u(V(Y)) - u(W(Y))) &= \omega_{J_1} \left(\sum_{\substack{m \in I \\ m \neq 0}} u_m ([V(Y)]^m - [W(Y)]^m) \right) \geq \\ &\geq \inf_{m \in I} \omega_{J_1}(v_i(Y) - w_i(Y)) = \omega_{J_1}(V(Y) - W(Y)), \end{aligned}$$

c'est-à dire la relation 3° .

3. Résolution des équations dans un anneau de séries formelles

Dans ce paragraphe nous donnons une nouvelle démonstration, en utilisant un procédé itératif, pour un résultat relatif à la résolution des équations dans un anneau de séries formelles [2].

Soit $U(X, Y) = (u_j(X, Y))_{j \in J}$, $u_j(X, Y) \in A[[X_i, Y_l]]_{i \in I, l \in J}$, $j \in J$. Il suit de [1, proposition 1] que

$$U(X, Y) = \sum_{m \in J} U_m(X) Y^m,$$

avec

$$U_m(X) = (u_{j,m}(X))_{j \in J}, \quad u_{j,m}(X) \in A[[X_i]]_{i \in I}, \quad j \in J.$$

Supposons que

$$(i) \quad \omega(U_0(X)) \geq 1.$$

$$(ii) \text{ La matrice } \mathcal{U} = (u_{k,e(j)}(X))_{k,j \in J}, \text{ est invertible.}$$

Alors on a

Proposition 5. *Si les conditions (i), (ii) sont vérifiées, alors il existe une série formelle vectorielle unique $V(X) \in A[[X_i]]_{i \in I}$, $\omega(V(X)) \geq 1$, telle que,*

$$(3.1) \quad U(X, V(X)) = 0.$$

En effet, il résulte de la condition (ii) qu'il existe une matrice $\mathcal{U}^{-1}$, telle que $\mathcal{U}^{-1}\mathcal{U} = E$, E — la matrice unité. Posons $W(X, Y) = \mathcal{U}^{-1}U(X, Y)$. Comme $U(X, Y) = \mathcal{U}W(X, Y)$, il résulte que la résolution de (1) est équivalente à la résolution de l'équation

$$(3.2) \quad W(X, V(X)) = 0.$$

Mais nous avons:

$$\begin{aligned} W(X, Y) &= \mathcal{U}^{-1}U(X, Y) = \mathcal{U}^{-1}U_0(X) + \sum_{\substack{m \in J \\ |m| \geq 2}} \mathcal{U}^{-1}U_m(X)Y^m = \\ &= -W_0(X) + EY - \sum_{m \in J, |m| \geq 2} W_m(X)Y^m. \end{aligned}$$

Comme $\omega(U_0(X)) \geq 1$ et $\mathcal{U}^{-1}$ est une matrice invertible, il résulte que $\omega(W_0(X)) \geq 1$. De cette manière la résolution de l'équation (2) est équivalente à résolution de l'équation:

$$(3.3) \quad V(X) = W_0(X) + \sum_{m \in J, |m| \geq 2} W_m(X)[V(X)]^m.$$

Utilisons la méthode des approximations successives. Soit $\{V^{(k)}(X)\}_{k \in \mathbb{N}}$, une suite de séries formelles définies par:

$$V^{k+1}(X) = W_0(X) + \sum_{m \in J, |m| \geq 2} W_m(X) [V^{(k)}(X)]^m; \quad k \in N, \quad V^{(0)}(X) = 0.$$

Alors

$$\begin{aligned} \omega(V^{k+1}(X) - V^{(k)}(X)) &= \omega\left(\sum_{\substack{m \in J \\ |m| \geq 2}} W_m(X) ([V^{(k)}(X)]^m - [V^{(k-1)}(X)]^m)\right) \geq \\ &\geq \omega\left(\sum_{\substack{m \in J \\ |m| = 2}} W_m(X) ([V^{(k)}(X)]^m - [V^{(k-1)}(X)]^m)\right) \geq \inf_{\substack{m \in J \\ |m| = 2}} \omega([V^{(k)}(X)]^m - [V^{(k-1)}(X)]^m) \geq \\ &\geq \inf_{i,j \in J} \omega(v_i^{(k)}(X) v_j^{(k)}(X) - v_i^{(k-1)}(X) v_j^{(k-1)}(X)) = \\ &= \inf_{i,j \in J} \omega(v_j^{(k)}(X) (v_i^{(k)}(X) - v_i^{(k-1)}(X)) + v_i^{(k-1)}(X) (v_j^{(k)}(X) - v_j^{(k-1)}(X))) \geq \\ &\geq \inf_{i,j \in J} \omega(v_j^{(k)}(X) (v_i^{(k)}(X) - v_i^{(k-1)}(X))) \geq \omega(V^{(k)}(X) - V^{(k-1)}(X)) + 1 \end{aligned}$$

pour tout k , ($k = 1, 2, \dots$); parce que $\omega(v_j^{(k)}(X)) \geq 1$.

Donc

$$\omega(V^{(k+1)}(X) - V^{(k)}(X)) \geq \omega(V^{(k)}(X) - V^{(k-1)}(X)) + 1.$$

Comme $\omega(V^{(1)}(X)) \geq 1$, il résulte que :

$$(3.4) \quad \omega(V^{(k+1)}(X) - V^{(k)}(X)) \geq k + 1.$$

Donc $\{V^{(k)}(X)\}_{k \in N}$ est une suite de Cauchy dans la topologie forte de $(A[[X_i]]_{i \in I})^J$ et par suite convergente. De la continuité de l'application de composition régulière dans la topologie forte, il résulte que la limite de cette suite satisfait à l'équation (3).

En supposant que $\bar{V}(X)$ est une autre solution de ce problème, il résulte comme plus haut que :

$$\omega(V(X) - \bar{V}(X)) \geq \omega(V(X) - \bar{V}(X)) + 1.$$

Donc

$$\omega(V(X) - \bar{V}(X)) = +\infty.$$

C'est-à-dire $V(X) = \bar{V}(X)$.

Il résulte de la relation (4) que :

$$\omega(V(X) - V^{(k)}(X)) \geq k + 1.$$

C'est-à-dire l'approximation de rang k , $V^{(k)}(X)$ contient la partie polynomiale de degré $k + 1$ de la solution exacte $V(X)$.

4. Les applications de substitution et de composition dans la topologie faible

1. Soit A un anneau topologique, commutatif, ayant élément unité, noté 1. La topologie faible de $A[[X_i]]_{i \in I}$, est la topologie produit dans l'ensemble A_I . Elle coïncide avec la topologie forte $T(\omega)$, si et seulement si l'anneau A est muni de la topologie discrète.

Proposition 6. *Les applications de substitution et de composition régulière sont continues si les anneaux de séries formelles respectives sont munis des topologies faibles.*

Soient $u(X) = \sum_{m \in \mathbf{I}} u_m X^m \in A[[X_i]]_{i \in I}$, et $V(Y) = (v_i(Y))_{i \in I}$,

$v_i(Y) \in A[[Y_j]]_{j \in J}$ avec $\omega_{J_1}(V) \geq 1$, $J_1 \subset J$. Alors

$$u(V(Y)) = \sum_{m \in \mathbf{I}} u_m [V(Y)]^m.$$

De la linéarité des projections canoniques,

$$p_r : A[[Y_j]]_{j \in J} \rightarrow A, \quad r \in J$$

et des définitions des opérations dans un anneau de séries formelles il résulte que :

$$\begin{aligned} p_r(u(V(Y))) &= \sum_{m \in \mathbf{I}} u_m p_r([V(Y)]^m) = \\ &= \sum_{\substack{m \in \mathbf{I} \\ |m| \leq |r|}} u_m \sum_{r(1) + \dots + r(n) = r} p_{r(1)}([v_1(Y)]^{m_1}) \dots p_{r(n)}([v_n(Y)]^{m_n}). \end{aligned}$$

Supposons que la série formelle $V(Y)$ est fixée (cas de l'application de substitution régulière) et $\omega_{J_1}(V) \geq 1$, $J_1 \subset J$. Alors

$$w_m = \sum_{r(1) + \dots + r(n) = m} p_{r(1)}([v_1(Y)]^{m_1}) \dots p_{r(n)}([v_n(Y)]^{m_n})$$

est un élément fixé de A .

Si p'_m , $m \in \mathbf{I}$, sont les projections canoniques de $A[[X_i]]_{i \in I}$ dans A , comme $u_m = p'_m(u)$, $m \in \mathbf{I}$, il résulte que :

$$p_r(u(V(Y))) = \sum_{\substack{m \in \mathbf{I} \\ |m| \leq |r|}} w_m p'_m(u(X)).$$

Donc

$$p_r \circ \Phi_V = \sum_{\substack{m \in \mathbf{I} \\ |m| \leq |r|}} w_m p'_m, \quad r \in J.$$

On en déduit que pour tout $r \in \mathbb{J}$, l'application $p_r \circ \phi_V$, s'exprime comme une combinaison linéaire finie de projections canoniques p'_m qui sont continues. Il résulte que $p_r \circ \phi_V$, $r \in \mathbb{J}$, sont continues dans la topologie faible, et donc l'application ϕ_V est continue.

Fixons maintenant la série formelle $u(X)$ (cas de l'application de composition). Soit P_i , $i \in I$, les projections canoniques :

$$P_i : (A[[Y_j]]_{j \in J})^I \rightarrow A[[Y_j]]_{j \in J}.$$

Donc pour tout

$$V(Y) = (v_i(Y))_{i \in I} \in (A[[Y_j]]_{j \in J})^I$$

nous avons $P_i(V) = v_i$, $i \in I$. Alors

$$p_r(u(V(Y))) = \sum_{\substack{m \in \mathbb{I} \\ |m| \leq |r|}} u_m \sum_{r^{(1)} + \dots + r^{(n)} = r} p_r^{(1)}([P_1(V)]^{m_1}) \dots p_r^{(n)}([P_n(V)]^{m_n}).$$

Donc

$$p_r \circ \phi_u = \sum_{\substack{m \in \mathbb{I} \\ |m| \leq |r|}} u_m \sum_{r^{(1)} + \dots + r^{(n)} = r} (p_r^{(1)} P_1^{m_1}) \dots (p_r^{(n)} P_n^{m_n}).$$

C'est-à dire pour tout $r \in \mathbb{J}$, l'application $p_r \circ \phi_u$, s'exprime comme une combinaison linéaire finie de produits finis d'applications p_r et P_i , qui sont continues, d'où la proposition.

2. La composition des séries formelles $u(X)$ et $V(Y)$ peut être définie dans la topologie faible, avec des conditions moins restrictives. Ainsi la composition de séries formelles $u(X)$ et $V(Y)$ est définie si et seulement si, la série avec le terme général $u_m [V(Y)]^m$ est convergente dans la topologie faible de $A[[Y_j]]_{j \in J}$. La somme de cette série se note aussi par $u(V(Y))$. Lorsque $\omega_J(V) \geq 1$, $J_1 \subset J$, cette condition est évidemment satisfaite. La proposition 3 est valable dans ce cas aussi.

Pour tout $u(X) = \sum_{m \in \mathbb{I}} u_m X^m \in A[[X_i]]_{i \in I}$, nous appelons *domaine de convergence*, noté $C(u)$, l'ensemble des éléments $x = (x_i)_{i \in I} \in A^I$, pour lesquels la série avec le terme général $u_{(m_i)} \prod_{i \in I} (x_i)^{m_i}$ est convergente dans l'anneau topologique A . Pour tout $x \in C(u)$, notons

$$u(x) = \sum_{(m_i) \in \mathbb{I}} u_{(m_i)} \prod_{i \in I} (x_i)^{m_i} = \sum_{m \in \mathbb{I}} u_m x^m.$$

Evidemment $0 \in C(u)$. Par exemple pour la série $u(X) = \sum_{m \in \mathbb{I}} X^m$, le domaine de convergence est formé de tous les éléments topologiques nilpotents dans A (c'est à dire ceux éléments pour lesquels $x^n \rightarrow 0$, $n \rightarrow \infty$).

De cette manière, pour toute série formelle $u(X)$, il existe une application : $\bar{u} : C(u) \rightarrow A$, définie par $\bar{u}(x) = u(x)$, pour tout $x \in C(u)$.

3. Proposition 7. Soient $u(X) \in A[[X_i]]_{i \in I}$ et $V(Y) = (v_i(Y))_{i \in I}$, $v_i(Y) \in A[[Y_j]]_{j \in J}$. Lorsque $V(0) \in C(D^m u)$, $m \in I$, alors la série avec le terme général $u_m [V(Y)]^m$, est convergente dans la topologie faible de $A[[Y_j]]_{j \in J}$, et donc la composition $u(V(Y))$ est définie.

Soit

$$W(Y) = u(V(Y)) = \sum_{m \in I} u_m [V(Y)]^m = \sum_{r \in J} \frac{1}{r!} D_Y^r W(0) Y^r.$$

Donc

$$(4.1) \quad u(V(Y)) = \sum_{r \in J} \frac{1}{r!} D_Y^r u(V(0)) Y^r$$

et donc

$$p_r(u(V(Y))) = \frac{1}{r!} D_Y^r u(V(0)), \quad r \in J.$$

Il résulte d'ici que $u(V(Y))$ existe quand les éléments $D_Y^r u(V(0))$ sont définies. D'après [1, relation (3.1)], $D_Y^r u(V(0))$ est définie s'il existe $D_X^m u(V(0))$, $m \in I$, $|m| \leq |r|$, considérées comme des sommes de séries convergentes dans A . Il s'ensuit de l'hypothèse $V(0) \in C(D_X^m u)$, que $D_Y^r u(V(0)) \in A$, et donc $u(V(Y)) \in A[[Y_j]]_{j \in J}$ est donnée par la relation (3.1).

5. La topologie faible de $A[[X_i]]_{i \in I}$.

Soit A un anneau valué complet non-discret.

Soit $J \subset I$, et $\bar{J} = J \cup \{0\}$ ($0 \in \bar{J}$). Considerons l'application :

$$\Phi_J : A[[X_i]]_{i \in I} \rightarrow R[[X_j]]_{j \in \bar{J}},$$

définie par

$$\Phi_J(u) = \sum_{m_J \in J} \sum_{k \in N} \frac{M_{k,m_J}}{k! m_J!} X_o^k X^{m_J}, \quad M_{k,m_J} = \sup_{|m|_L = k} |u_m|, \quad L = I - J$$

pour tout

$$u(X) = \sum_{m \in I} \frac{u_m}{m!} X^m \in A[[X_i]]_{i \in I}.$$

Remarquons que l'algèbre $R[[X_j]]_{j \in \bar{J}}$, est une algèbre ordonnée par la relation : $u \leq v$ si $p_m(u) \leq p_m(v)$, $m \in \bar{J}$.

En utilisant la définition de l'application Φ_J , il résulte par un calcul élémentaire la proposition suivante :

Proposition 8. *L'application Φ_J , a les propriétés suivantes :*

$$1^\circ \Phi_J(u) \geq 0; \quad \Phi_J(u) = 0 \Leftrightarrow u = 0;$$

$$2^\circ \Phi_J(u + v) \leq \Phi_J(u) + \Phi_J(v);$$

$$3^\circ \Phi_J(xu) = |x| \Phi_J(u), \quad x \in A;$$

$$4^\circ \Phi_J(uv) \leq \Phi_J(u) \cdot \Phi_J(v)$$

pour tout $u, v \in A[[X_i]]_{i \in I}$.

Cette proposition entraîne que $\rho(u, v) = \Phi_J(u - v)$ considéré comme application de $A[[X_i]]_{i \in I} \times A[[X_i]]_{i \in I}$, dans l'algèbre ordonnée $R[[X_j]]_{j \in J}$, est une pseudométrique sur $A[[X_i]]_{i \in I}$ [4, Chap. 3, § 3.3].

Nous avons

$$\begin{aligned} \rho(u, v) &= 0 \Leftrightarrow u = v, \\ \rho(u, v) &\leq \rho(u, w) + \rho(w, v), \\ \rho(u, v) &= \rho(v, u). \end{aligned}$$

Nous dirons que $\lim u^{(k)}(X) = u(X)$ si $\lim \rho(u^{(k)}, u) = 0$, l'algèbre $R[[X_j]]_{j \in J}$ étant munie de la topologie faible. On peut démontrer que cette notion de convergence, conduit à la topologie faible et réciproquement. Donc :

Proposition 9. *Soit A un anneau valué non-discret. Alors la topologie faible de $A[[X_i]]_{i \in I}$ est une topologie d'espace pseudométrique, avec $\rho(u, v) = \Phi_J(u - v)$, $J \subset I$. Lorsque A est un anneau topologique complet alors $A[[X_i]]_{i \in I}$, est un espace pseudométrique complet.*

Proposition 10. *Pour tout $r \in I$, nous avons*

$$\Phi_J(D^r u) \leq D_0^{r|L} D^{r_J} \Phi_J(u); \quad D_0 = D_{X_0}.$$

En effet, soit

$$u(X) = \sum_{m \in I} \frac{u_m}{m!} X^m.$$

Alors

$$D^r u(X) = \sum_{m \in I} \frac{(m+r)!}{m!} p_{m+r}(u) = \sum_{m \in I} \frac{u_{m+r}}{m!} X^m.$$

Donc

$$\Phi_J(D^r u) = \sum_{m_J \in J} \sum_{k \in N} \frac{M'_{k, m_J}}{k! m_J!} X_0^k X^{m_J}$$

et

$$M'_{k, m_J} = \sup_{m_L = k} |u_{m+r}| \leq M_{k+|r|L, m_J+r_J}.$$

Mais

$$D_0^{r|L} D^{r_J} \Phi_J(u) = \sum_{m_J \in J} \sum_{k \in N} \frac{M_{k+|r|L, m_J+r_J}}{k! m_J!} X_0^k X^{m_J},$$

d'où la proposition.

Il résulte de cette proposition que,

$$\rho(D^r u, D^r v) \leq P\rho(u, v),$$

avec $P = D_0^{|\mathbf{r}|L} D^{\mathbf{r}J}$, opérateur positivement défini dans $R[[X_i]]_{i \in J}$. Il résulte d'ici que l'opérateur de dérivation partielle D^r est borné dans l'espace pseudométrique $(A[[X_i]]_{i \in I}, \rho)$ et donc continu [4, Chap. 6, § 6.1].

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ASUPRA UNOR STRUCTURI TOPOLOGICE ÎNTR-UN INEL DE SERII FORMALE (II)

(Rezumat)

În [1], după ce s-au studiat topologiile ordinelor definite, s-au introdus aplicațiile de substituție φ_v și de compoziție regulată ψ_u și s-a cercutat imaginea compoziției regulate a două serii prin aplicația de derivare parțială D^r . În nota de față se studiază aplicațiile φ_v și ψ_u în topologia tare și în topologia slabă. Se dă o nouă demonstrație pentru un rezultat relativ la rezolvarea ecuațiilor într-un inel de serii formale și se studiază topologia slabă în $A[[X_i]]_{i \in I}$, A fiind un inel topologic nediscret. Se arată că în acest caz $A[[X_i]]_{i \in I}$ este un spațiu pseudometric (propoziția 9) și că aplicația D^r este mărginită (propoziția 10).

THE SET OF FIXED POINTS FOR LINEAR MULTIFUNCTIONS

BY

AL. SABADIȘ

Let E and F be two topological linear spaces and $\Gamma \subset E \times P(F)$ a multifunction defined on E with the range F .

Definition 1. Γ is called linear if $y_1 \in \Gamma(x_1)$ and $y_2 \in \Gamma(x_2) \Rightarrow y_1 + y_2 \in \Gamma(x_1 + x_2)$ and $\forall \alpha \in R \wedge y \in \Gamma(x) \Rightarrow \alpha y \in \Gamma(\alpha x)$.

Definition 2. Γ is called point compact if $\forall x \in E$, $\Gamma(x)$ is compact in F .

Definition 3. Γ is called upper semi-continuous on E if $\forall x \in E$ with $\Gamma(x) \subset D$, $D \in \tau_F$ there is a neighbourhood of x_0 , $V(x_0) \subset E$ with $\Gamma(V(x_0)) \subset D$.

Definition 4. Γ is closed on E if, $\forall x \in E$ and $y \in F$ with $y \in \overline{\Gamma(x_0)}$ there are two neighbourhoods $U(x_0) \subset E$ and $V(y_0) \subset F$ so that $\Gamma(x) \cap V(y_0) = \emptyset$, $\forall x \in U(x_0)$.

Lemma 1. If E is a topological linear space and Γ is linear, then $\Gamma(E) = \bigcup_{x \in E} \Gamma(x)$ is a linear subspace of F .

Proof. Let $y_1 \in \Gamma(E)$ and $y_2 \in \Gamma(E)$. Then there will be $x_1 \in E$ and $x_2 \in E$ so that $y_1 \in \Gamma(x_1)$ and $y_2 \in \Gamma(x_2)$.

Since Γ is linear it follows that $y_1 + y_2 \in \Gamma(x_1 + x_2) \subset \Gamma(E)$.

On the other hand $\forall \alpha \in R$ and $\forall y \in \Gamma(E)$ we have $\alpha y \in \Gamma(\alpha x) \subset \Gamma(E)$ where $x \in E$ and $y \in \Gamma(x)$ and the lemma is proved.

Lemma 2. If $\Gamma \subset E \times P(E)$, E compact and $\Gamma(x) \neq \emptyset$, $\forall x \in E$, and if Γ is upper semi-continuous and point compact, then the multifunction Γ , admits only one compact $K \subset E$ with $\Gamma(K) = K$.

Proof. A. If $\Gamma(E) = E$ then the lemma is proved where $K = X$.

B. If $\Gamma(E) \neq E$ then $E \neq \Gamma(E) \neq \Gamma^2(E) \neq \dots \neq \Gamma^n(E) \neq \dots$ and $E \supset \Gamma(E) \supset \Gamma^2(E) \supset \dots \supset \Gamma^n(E) \supset \dots$. Because $\Gamma^n(E)$ is compact, $\forall n \in N$,

we have $K = \bigcap_{n=1}^{\infty} \Gamma^n(E) \neq \emptyset$. Therefore, $\forall n \in N, K \subset \Gamma^{n-1}(E) \Rightarrow \Gamma(K) \subset \Gamma^n(E)$. It follows that $\Gamma(K) \subset \bigcap_{n=1}^{\infty} \Gamma^n(E) = K$.

We shall show that $K \subset \Gamma(K)$. Let $x_0 \in K$ be fixed. Then, $\forall n \in N, \exists x_n \in \Gamma^n(E)$ and $x_0 \in \Gamma^n(E)$. Therefore, x_0 is an adherent point for the succession $\{x_n\}$. Then there is a subsuccession $\{x_{k_n}\} \subset \{x_n\}$ so that $x_{k_n} \rightarrow x_0$. Let be $x \in K$. We have $(x_{k_n}, x) \rightarrow (x_0, x) \in \bigcup_{x \in E} \Gamma(x)$ for an upper semi-continuous multifunction and point compact is closed. Therefore $x \in \Gamma(x_0) \subset \Gamma(K)$ and the lemma is proved. It follows from the proof that the compact K is maximum and unique.

Corollary. *If in the lemma 2, the space E is compact and Γ is a linear multifunction, then K is a subspace of E .*

Proof. $\Gamma^n(E)$ is a subspace of $E, \forall n \in N$. It follows that K is also a subspace of E .

Theorem 1. *If E is a locally convex space, $C \subset E$ is convex and compact, and function $f: C \rightarrow C$ a linear continuous function, then there is $x_0 \in C$ with $f(x_0) = x_0$ and $x_0 \in C \cap (\bigcap_{n=1}^{\infty} \Gamma^n(E))$.*

Proof. Since f is linear and continuous, we can apply the second lemma. Therefore there is one and only one compact $K \subset E$ with

$$f(K) = K, \quad K = \bigcap_{n=1}^{\infty} \Gamma^n(E).$$

Since K is one linear subspace of E and C is a convex compact subset of E , it follows that $K \cup C$ is a convex compact set. We have $f(K \cap C) \subset f(K) \cap f(C) = K \cap f(C) \subset K \cap C$, and $K \cap C$ is a convex compact set. Therefore $f: K \cap C \rightarrow K \cap C$.

According to the **Schauder-Tihonov** theorem it results that there is a fixed point set $P, P \subset K \cap C = (\bigcap_{n=1}^{\infty} \Gamma^n(E)) \cap C$.

We shall show that P is a closed subset of $K \cap C$ and, therefore, it is compact. Let $\{x_n\} \subset P$ be a successive, i. e. $f(x_n) = x_n$. We put $\lim_n x_n = x^*$ and we have :

$$\left. \begin{aligned} \lim_n f(x_n) &= f(\lim_n x_n) = f(x^*) \\ \lim_n f(x_n) &= \lim_n x_n = x^* \end{aligned} \right\} \Rightarrow f(x^*) = x^* \Rightarrow x^* \in P.$$

Therefore, P is a compact subset of $K \cap C$ and the theorem is proved.

Theorem 2. *If E is a locally, convex space, $C \subset E$ a convex compact subset for it and $\Gamma: C \rightarrow C$ a linear upper semi-continuous point con-*

convex and point compact multifunction with $\Gamma(x) \neq \emptyset$, $\forall x \in C$, then Γ has a fixed point set P , $P \subset C \cap (\bigcap_{n=1}^{\infty} \Gamma_n(E))$.

Proof. According to lemma 2, there is one maximum compact $K \subset E$ with $\Gamma(K) = K$ and $K = \bigcap_{n=1}^{\infty} \Gamma^n(E)$. Therefore $\Gamma(K \cap C) \subset \Gamma(K) \cap \Gamma(C) = K \cap \Gamma(C) \subset K \cap C$.

Since $K \cap C$ is a convex compact set, from the Ky Fan fixed point theorem, there is a nonempty fixed point set of Γ , P , $P \subset K \cap C$. We show that P is a closed subset of $K \cap C$. Let $\{x_n\}$ be a succession, $\{x_n\} \subset P$, i. e. $x_n \in \Gamma(x_n)$ and $x_n \rightarrow x^*$. Then by proposition 2.5 from [1], it results $x^* \in \Gamma(x^*)$, i. e. $x^* \in P$.

Therefore P is a compact subset of $K \cap C$ and the theorem is proved.

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MULȚIMEA PUNCTELOR FIXE PENTRU MULTIFUNCȚII LINIARE

(Rezumat)

Se dau teoreme în legătură cu mulțimea punctelor fixe a unei multifuncții (funcții) liniare și superior semicontinue.

SPACE DERIVATIVES (I)¹⁾

BY

ZORAN VRCELJ

1. Introduction

Space derivatives are known to be unavoidable quantities in vector analysis and in almost all its applications throughout physics and engineering. Space derivatives were systematically treated (probably for the first time) by Jung [1] in 1908. But some methods of vector analysis directly related to the obtaining of all results concerning space derivatives had been earlier developed. Moreover, the main representatives of space derivatives — gradient, divergence, and curl — came to the knowledge in the course of the 19th century. The evaluation of coordinate representations of polar and axial derivatives from their integral definitions is based on the assumption of an infinitesimal cubical, cylindrical or spherical volume element and an infinitesimal rectangular or circular surface element, respectively. An infinitesimal cubical volume element was used by Heaviside [2], Gibbs [3], Föppl [4], and in many well-known textbooks on vector analysis [5]...[7]; an infinitesimal cylindrical volume element was applied by Jung [1] as well as in a number of standard books on the subject [5]...[9], while an infinitesimal rectangular surface element was employed by Heaviside [2], Föppl [4], and in some textbooks [7]. An interpretation of divergence and curl in terms of vector growth using an infinitesimal spherical volume element and an infinitesimal circular surface element, respectively, was obtained more than half a century ago [10]. The integral definitions of polar and axial derivatives are directly related to Gauss's and Stokes's theorems, respectively; moreover, they could be comprehended as local analogues of the integral theorems [11]. Gauss's theorem has been derived for a tensor field using the integral representation of the operator nabla and of the idemfactor in three-dimensional Euclidean space [12]. The divergence operator in rectangular coordinates has been obtained using an infinitesimal spherical volume element and connecting surface integrals to surface averages [13].

The well-known assumption usually made in all demonstrations regarding space derivatives is the continuity of the first-order partial derivatives or the "continuous differentiability to the order required by the context" [12]. As it is also well-known, the continuity of the first-order partial derivatives is a sufficient, but not necessary condition for the differentiability of a function in several variables. It is believed that most of the results which have been proved under continuous differentiability condition hold under weaker conditions, too.

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The gradient of a scalar field can be defined on the basis of the differentiability of functionals [7] and its equality to the polar derivative (integral definition) has been proved [14]. But the method presented in [14] requires some modifications in order to be applicable to vector fields. In [13] curl is not dealt with, while the procedure of [12] cannot be simply extended to the case of Stokes's theorem.

In this paper the differentiability is proved to be the essential condition for the existence of all space derivatives. The method presented here is applicable both to scalar and vector fields and the results are manifestly independent of the orientation of coordinates. Considerations are restricted to three-dimensional Euclidean spaces.

2. Differentiability Property

Let $\mathfrak{F}$ be a scalar or vector field (scalar or vector point function) defined in some region of three-dimensional Euclidean space, and P — an arbitrary fixed point contained in this region.

Directional derivative of $\mathfrak{F}$ at P in the direction of the unit vector $\mathbf{n}$ is denoted by $D_{\mathbf{n}}\mathfrak{F}(P)$. Directional derivative is the simplest type of space derivatives (cf. „Linearableitung“ in [1] and „derivada direccional particular“ in [15]); one-dimensional vicinity of P (a segment of straight line containing P) is involved in the limiting process.

Differential of $\mathfrak{F}$ at P is denoted by $\mathfrak{D}(P, \mathbf{rn})$, where $\mathbf{rn}$ is the position vector of a point M contained in a vicinity of the point P with respect to this point as an origin (cf. the Fréchet derivative in [16]).

The field $\mathfrak{F}$ is *differentiable* at the point P if and only if its increment can be represented (in the unique way) by the expression

$$(1) \quad \mathfrak{F}(M) - \mathfrak{F}(P) = \mathfrak{D}(P, \mathbf{rn}) + r\mathfrak{Q}(P, M),$$

where $\mathfrak{D}(P, \mathbf{rn})$ is a bounded linear functional in the variable $\mathbf{rn}$ and $\mathfrak{Q}(P, M) \rightarrow 0$ when $r \rightarrow 0$.

This definition implies that the differentiability of $\mathfrak{F}$ at P is a sufficient condition for the existence of the directional derivative $D_{\mathbf{n}}\mathfrak{F}(P)$ in every direction $\mathbf{n}$. Then the equality

$$(2) \quad \mathfrak{D}(P, \mathbf{n}) = D_{\mathbf{n}}\mathfrak{F}(P)$$

holds for arbitrary $\mathbf{n}$.

In the case of orthonormal basis $(\mathbf{e}_i)$ ($i = 1, 2, 3$) using Eq. (2) the following expression is obtained

$$(3) \quad D_{\mathbf{n}}\mathfrak{F}(P) = \sum_{i=1}^3 (\mathbf{n} \cdot \mathbf{e}_i) D_{\mathbf{e}_i}\mathfrak{F}(P)$$

yielding the relation

$$(4) \quad \mathfrak{F}(M) - \mathfrak{F}(P) = r \sum_{i=1}^3 (\mathbf{n} \cdot \mathbf{e}_i) D_{\mathbf{e}_i}\mathfrak{F}(P) + r\mathfrak{Q}(P, M),$$

of basic importance in the demonstrations from the second part of this work. The differentiability of $\mathfrak{F}$ at P is evidently a sufficient condition for the validity of the representation (4).

The following simple *lemma* will be also used in future: *If $\mathfrak{F}$ is continuous in a vicinity of P and differentiable at P , then Ω is continuous in this vicinity.*

3. Integral Definition

Consider a volume element V bounded by a smooth closed surface Σ so that $P \in V$. Let $d\sigma$ be a surface element of Σ , $\mathbf{n}$ —the unit outer normal vector field to the surface and M —a point on the surface. The vector surface integral of the field $\mathfrak{F}$ over the surface Σ is an additive function of the enclosed region V . The general derivative of an additive set function in the special cases which are of interest in vector analysis yields quantities defined by

$$(5) \quad \nabla \circ \mathfrak{F}(P) \stackrel{\text{def}}{=} \lim_{V \rightarrow P} \frac{1}{V} \oint_{\Sigma} \mathbf{n} \circ \mathfrak{F}(M) d\sigma,$$

where the symbol ($\circ$) denotes ordinary (arithmetic), scalar (dot), or vector (cross) multiplication. These quantities are determined by the properties of the given field in an infinitesimal vicinity of the point P , independently of any preferable direction. Therefore, we are referred to the quantities defined by Eq. (5) as *polar derivatives* at the point P (cf. „Polarableitung“ in [1] and „isotropic invariant“ in [11]).

Now a plane through the point P given by the unit normal vector $\mathbf{s}$ can be introduced as a two-dimensional flat subspace of three-dimensional Euclidean space. Consider a plane surface element S bounded by a smooth closed curve Λ in the given plane so that $P \in S$. Let $d\lambda$ be a line element of Λ , $\mathbf{t}$ —the unit tangent vector field to the curve, and M —a point on the curve. The vectors $\mathbf{s}$, $\overrightarrow{PM}$, $\mathbf{t}$ form a right-handed system. The vector line integral of the field $\mathfrak{F}$ along the curve Λ is an additive function of the enclosed region S . The general derivative in the special cases which are of interest here gives quantities defined by

$$(6) \quad \nabla_s \circ \mathfrak{F}(P) \stackrel{\text{def}}{=} \lim_{S \rightarrow P} \frac{1}{S} \oint_{\Lambda} \mathbf{t} \circ \mathfrak{F}(M) d\lambda.$$

These quantities are determined by the properties of the given field in an infinitesimal vicinity contained in the given plane through P , i. e. they depend on the direction of $\mathbf{s}$. Therefore, we are referred to the quantities defined by Eq. (6) as *axial derivatives* at the point P in the direction of $\mathbf{s}$ (cf. „Axialableitung“ in [1] and „anisotropic invariant“ in [11]).

4. Some Surface and Line Integrals

Let Σ be a *spherical* surface of radius r , $\mathbf{n}$ — the unit outer normal vector field to the surface, and $\mathbf{c}, \mathbf{c}_1, \mathbf{c}_2$ — arbitrary constant vectors. Then the following results are valid:

$$(7) \quad \oint_{\Sigma} (\mathbf{n} \cdot \mathbf{c}) \mathbf{n} d\sigma = \frac{4}{3} \pi r^2 \mathbf{c},$$

$$(8) \quad \oint_{\Sigma} (\mathbf{n} \cdot \mathbf{c}_1) (\mathbf{n} \cdot \mathbf{c}_2) d\sigma = \frac{4}{3} \pi r^2 \mathbf{c}_1 \cdot \mathbf{c}_2,$$

$$(9) \quad \oint_{\Sigma} (\mathbf{n} \cdot \mathbf{c}_1) (\mathbf{n} \times \mathbf{c}_2) d\sigma = \frac{4}{3} \pi r^2 \mathbf{c}_1 \times \mathbf{c}_2.$$

Let Λ be a *circular* curve of radius r , $\mathbf{t}$ — the unit tangent vector field to the curve, and $\mathbf{c}, \mathbf{c}_1, \mathbf{c}_2$ — arbitrary constant vectors ($\mathbf{c}$ lying in the plane of the curve). Then the following results hold:

$$(10) \quad \oint_{\Lambda} (\mathbf{t} \cdot \mathbf{c}) \mathbf{t} d\lambda = \pi r \mathbf{c},$$

$$(11) \quad \oint_{\Lambda} (\mathbf{t} \cdot \mathbf{c}_1) (\mathbf{t} \cdot \mathbf{c}_2) d\lambda = \pi r \mathbf{c}_1 \cdot \mathbf{c}_2,$$

$$(12) \quad \oint_{\Lambda} (\mathbf{t} \cdot \mathbf{c}_1) (\mathbf{t} \times \mathbf{c}_2) d\lambda = \pi r \mathbf{c}_1 \times \mathbf{c}_2.$$

The results (7)...(12) can be evaluated using spherical and polar coordinates respectively.

In the second part this results are to be applied to scalar and vector fields.

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DERIVATE SPAȚIALE (I)

(Rezumat)

Derivabilitatea într-un punct și continuitatea într-o vecinătate a unui câmp sînt condiții suficiente pentru existența tuturor derivatelor spațiale ale cîmpului în punct. Demonstrațiile se efectuează utilizînd proprietăți de derivabilitate și considerînd un volum infinitesimal sferic și un element de arie circular. Se obține o reprezentare a derivatelor polare și axiale prin intermediul derivatelor direcționale și se stabilesc unele relații de legătură.

NUMERICAL SOLUTION OF A BOUNDARY VALUE PROBLEM FREQUENTLY OCCURRING IN THE DESIGN OF CERTAIN ELECTRICAL NETWORKS (I) ¹⁾

BY

K. V. LEUNG, D. MANGERON²⁾ and M. N. OĞUZTÖRELI

I. Let $A = A(x)$ be a given 4×4 matrix function with elements $A_{ij} = A_{ij}(x)$ defined and sufficiently smooth in the interval $0 \leq x \leq 1$, $y = y(x)$ be a 4-dimensional column vector function with components $y_i = y_i(x)$, and a be a given 4-dimensional constant column vector with components a_i , ($i, j = 1, 2, 3, 4$). Consider the boundary value problem

$$(I.1) \quad \begin{cases} \frac{dy}{dx} = Ay + a, \\ y_1(0) = y_1(1) = 0, \quad y_2(0) = y_2(1) = 0. \end{cases}$$

Boundary value problems of this form frequently occur in the design of certain electrical networks. In this paper we deal only with the numerical solution of Eqs. (I.1). In the next Notes ³⁾ we shall present an analytical investigation of the same problem, and a computer simulation for an electrical network described by a particular boundary value problem of the form (I.1). Relevant literature is listed at the end of the paper [1]... [26].

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²⁾ The author wishes to express his sincere thanks to the University of Alberta and to Sir George Williams University of Montreal for the invaluable conditions offered to him to develop his work in the Department of Mathematics and in the Department of Computer Science of these universities, respectively.

³⁾ The main results of this set will be reported to the Naples Academy of Mathematical and Physical Sciences.

II. Assume that Eqs. (I.1) admit a solution. Let us denote by u and v the initial values of the functions y_3 and y_4 :

$$(II.1) \quad y_3(0) = u, \quad y_4(0) = v.$$

If the values of u and v are known, the boundary value problem (I.1) can be treated as an initial value problem of the following form:

$$(II.2) \quad \frac{dy}{dx} = Ay + a, \quad y_1(0) = y_2(0) = 0, \quad y_3(0) = u, \quad y_4(0) = v.$$

Because of the linearity of the differential system and the smoothness of the matrix $A(x)$, this initial value problem can be solved uniquely. Obviously, the values $y_1(1)$ and $y_2(1)$ will depend on u and v . Let $y_1(1) = F(u, v)$, $y_2(1) = G(u, v)$. We have to determine u and v such that

$$(II.3) \quad F(u, v) = 0, \quad G(u, v) = 0.$$

In the next sections we shall present a numerical scheme for the solutions of Eqs. (II.2)–(II.3). For the solution of Eqs. (II.2) we shall use Taylor series expansion with one step method with variable step size and variable order of approximations.

Before we start our analysis, let us subdivide the interval $0 \leq x \leq 1$ into N equal subinterval. Let $h = \frac{1}{N}$. We shall use the following notation:

$$(II.4) \quad y_r = y(rh), \quad A^{(n)} = \frac{d^n}{dx^n} A, \quad y_r^{(n)} = \left. \frac{d^n y}{dx^n} \right|_{x=rh}, \quad y_r^{(0)} = y_r,$$

for $r = 0, 1, \dots, N$ and $n = 0, 1, 2, \dots$.

III. We begin with the computation of the higher order derivatives of $y(x)$. Given $dy/dx = Ay + a$, we can easily show by mathematical inductions that

$$(III.1) \quad \frac{d^{n+1}y}{dx^{n+1}} = A_{(n)}y + a_{(n)},$$

where $A_{(0)} = A$, $a_{(0)} = a$ and

$$(III.2) \quad A_{(n)} = A_{(n-1)}^{(1)} + A_{(n-1)}A, \quad a_{(n)} = a_{(n-1)}^{(1)} + A_{(n-1)}a, \quad (n = 1, 2, 3, \dots),$$

and $a_{(n-1)}^{(1)} = da_{(n-1)}/dx$. The matrices $A_{(n)}$ can be expressed in terms of A , $A^{(1)}$, $A^{(2)}$, ..., $A^{(n)}$. Indeed, we have

$$(III.3) \quad \left\{ \begin{aligned} A_{(1)} &= A^{(1)} + AA, \\ A_{(2)} &= A^{(2)} + 2A^{(1)}A + AA^{(1)} + AAA, \\ A_{(3)} &= A^{(3)} + 3A^{(2)}A + 3A^{(1)}A^{(1)} + AA^{(2)} + 3A^{(1)}AA + \\ &\quad + 2AA^{(1)}A + AAA^{(1)} + AAAA, \\ A_{(4)} &= A^{(4)} + 4A^{(3)} + 6A^{(2)}A^{(1)} + 4A^{(1)}A^{(2)} + AA^{(3)} + \\ &\quad + 6A^{(2)}AA + 8A^{(1)}A^{(1)}A + 3AA^{(2)}A + 4A^{(1)}AA^{(1)} + \\ &\quad + 3AA^{(1)}A^{(1)} + AAA^{(1)} + 4A^{(1)}AAA + 3AA^{(1)}AA + \\ &\quad + 2AAA^{(1)}A + AAAA^{(1)} + AAAAA, \\ &\dots \end{aligned} \right.$$

Generally we have

$$(III.4) \quad A_{(k)} = \sum_{k=1}^{x+1} \left(\prod_{r \in U(x, k)} \alpha(r, x, k) A^{(r_1)} A^{(r_2)} \dots A^{(r_k)} \right),$$

where

$$(III.5) \quad U(x, k) = \left\{ r \mid \sum_{i=1}^k r_i = x - k + 1 \right\}$$

and

$$(III.6) \quad \alpha(r, x, k) = \alpha_1(r, x, k) + \alpha_2(r, x, k),$$

where

$$(III.7) \quad \alpha_1(r, x, k) = \begin{cases} 0 & \text{if } r_k \neq 0 \\ \alpha(r', x-1, k') & \text{if } k' = k-1 \text{ and } r_i = r'_i \\ & (i = 1, \dots, k-1) \end{cases}$$

and

$$(III.8) \quad \alpha_2(r, x, k) = \begin{cases} \sum_{k'} \alpha(r', x-1, k') & \text{if } k' = k \text{ and for all } r \text{ such that} \\ r_i^0 = r'_i + \delta_{i,r} & \text{for } 1 \leq r \leq k = k', \quad (i = 1, \dots, k) \\ 0, & \text{otherwise} \end{cases}$$

where r is a k -vector whose components are non-negative integers, and $\delta_{i,r}$ is the Kronecker delta.

IV. We now deal with the Taylor expansions. Clearly, we have

$$(IV.1) \quad y_{r+1} = y_r + h y_r^{(1)} + \frac{h^2}{2!} y_r^{(2)} + \dots + \frac{h^M}{M!} y_r^{(M)}$$

if

$$(IV.2) \quad \left\| \frac{h^{M+1}}{(M+1)!} y^{(M+1)} \right\| < \varepsilon,$$

where $\| \cdot \|$ denotes maximum vector norm. Hence

$$(IV.3) \quad y_{r+1} = B_{(r)} y_r + b_{(r)},$$

where

$$(IV.4) \quad \begin{cases} B_{(r)} = \left(1 + hA + \frac{h^2}{2!} A_{(1)} + \dots + \frac{h^M}{M!} A_{(M-1)} \right)_r, \\ b_{(r)} = \left(ha + \frac{h^2}{2!} a_{(1)} + \frac{h^3}{3!} a_{(3)} + \dots + \frac{h^M}{M!} a_{(M-1)} \right)_r. \end{cases}$$

Hence

$$(IV.5) \quad \begin{cases} y_1 = B_{(0)} y_0 + b_{(0)}, \\ y_2 = B_{(1)} y_1 + b_{(1)} = B_{(1)} B_{(0)} y_0 + B_{(1)} b_{(0)} + b_{(1)}, \\ y_3 = B_{(2)} y_2 + b_{(2)} = B_{(2)} B_{(1)} B_{(0)} y_0 + B_{(2)} B_{(1)} b_{(0)} + B_{(2)} b_{(1)} + b_{(2)}, \\ \dots \\ y_N = B_{(N-1)} B_{(N-2)} \dots B_{(3)} B_{(2)} B_{(1)} B_{(0)} y_0 + \\ + B_{(N-1)} B_{(N-2)} \dots B_{(3)} B_{(2)} B_{(1)} b_{(0)} + \\ + B_{(N-1)} B_{(N-2)} \dots B_{(3)} B_{(2)} b_{(1)} + \\ + \dots + \\ + B_{(N-1)} B_{(N-2)} b_{(N-3)} + \\ + B_{(N-1)} b_{(N-2)} + \\ + b_{(N-1)} = \\ = Ky_0 + k, \end{cases}$$

where

$$(IV.6) \quad K = B_{(N-1)} B_{(N-2)} B_{(N-3)} \dots B_{(2)} B_{(1)} B_{(0)}$$

and

$$(IV.7) \quad \begin{aligned} k = & B_{(N-1)} B_{(N-2)} \dots B_{(1)} b_{(0)} + \\ & + B_{(N-1)} B_{(N-2)} \dots B_{(2)} b_{(1)} + \\ & + \dots + \\ & + B_{(N-1)} B_{(N-2)} b_{(N-3)} + \\ & + B_{(N-1)} b_{(N-2)} + \\ & + b_{(N-1)}. \end{aligned}$$

Let us note that K and k do not depend on y_0 . We now write the equation $y_N = Ky_0 + k$ explicitly.

$$(IV.8) \quad \begin{bmatrix} y_{1,N} \\ y_{2,N} \\ y_{3,N} \\ y_{4,N} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ y_{3,0} \\ y_{4,0} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix},$$

since $y_{1,0} = y_{2,0} = 0$. Therefore

$$(IV.9) \quad \begin{cases} y_{1,N} = k_{13}y_{3,0} + k_{14}y_{4,0} + k_1, \\ y_{2,N} = k_{23}y_{3,0} + k_{24}y_{4,0} + k_2. \end{cases}$$

Since $y_{1,N} = y_{2,N} = 0$ and $y_{3,0} = u$, $y_{4,0} = v$, we have

$$(IV.10) \quad \begin{cases} k_{13}u + k_{14}v + k_1 = 0, \\ k_{23}u + k_{24}v + k_2 = 0. \end{cases}$$

System (IV.10) admits a unique solution if

$$(IV.11) \quad \Delta \equiv \begin{vmatrix} k_{13} & k_{14} \\ k_{23} & k_{24} \end{vmatrix} \neq 0.$$

In this case

$$(IV.12) \quad u = \frac{\begin{vmatrix} k_{14} & k_1 \\ k_{23} & k_2 \end{vmatrix}}{\Delta}, \quad v = \frac{\begin{vmatrix} k_1 & k_{13} \\ k_2 & k_{14} \end{vmatrix}}{\Delta},$$

and the boundary value problem (I.1) is completely solved using these values of u and v in the Eqs. (II.2).

Clearly, if $\Delta = 0$, Eqs. (IV.10) admit infinitely many solutions if

$$(IV.13) \quad \frac{k_{13}}{k_{23}} = \frac{k_{14}}{k_{24}} = \frac{k_1}{k_2}.$$

Further, Eqs. (IV.10) are not solvable if $\Delta = 0$ but Eqs. (IV.13) do not hold. In this case the boundary value problem (I.1) has no solution.

V. The flow hart of the above described numerical scheme is given below (Figs 1 and 2). The program listings in FORTRAN language and computer simulations with illustrative examples based on certain realistic electrical networks will be presented in the next Note of this series.

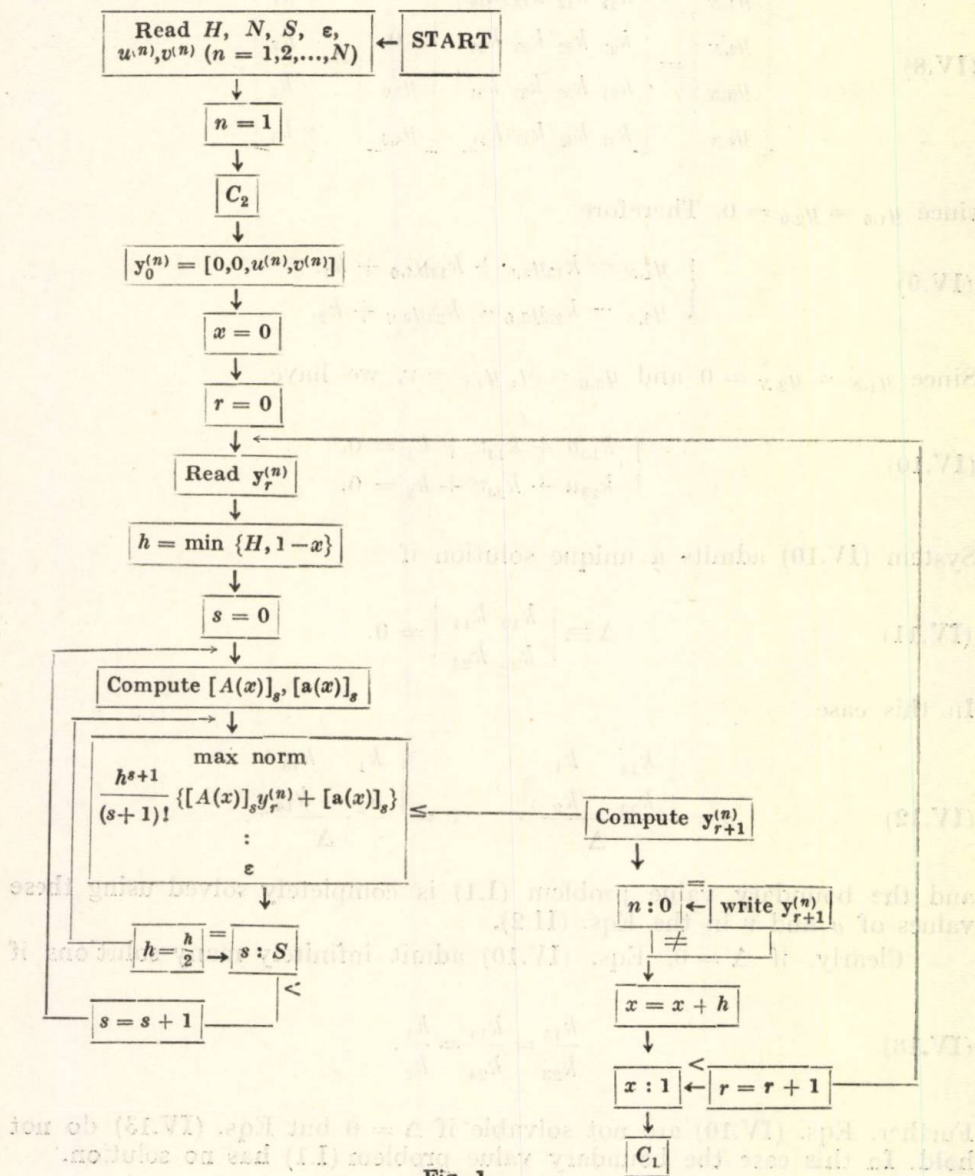


Fig. 1

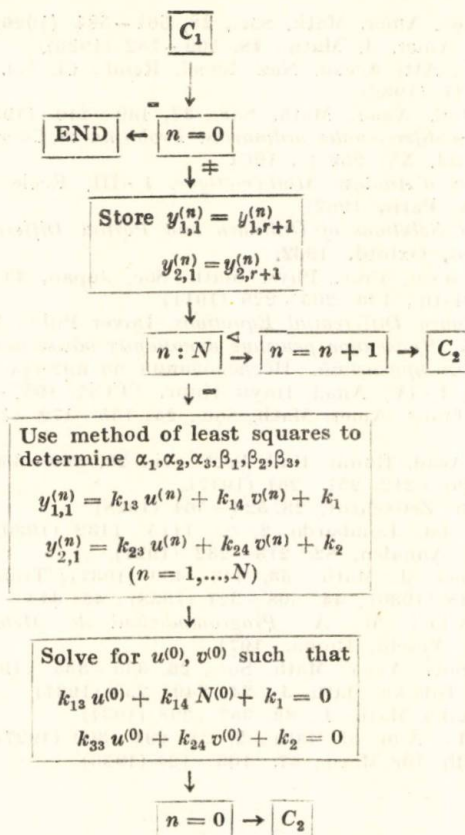


Fig. 2

Clearly, the numerical scheme presented in this paper can be used without any difficulty to the first order differential systems with $2n$ equations when n of the unknown functions are subjected to two-point boundary conditions.

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SOLUȚIE NUMERICĂ PENTRU O PROBLEMĂ PE CONTUR
CARE INTERVINE FRECVENT ÎN PROIECTAREA
UNOR REȚELE ELECTRICE (I)

(Rezumat)

Se dă soluția numerică a unei probleme pe contur pentru un sistem de patru ecuații diferențiale ordinare de ordinul întâi, care modelează anumite rețele electrice. În note viitoare se vor prezenta soluția analitică a problemei și o simulare pe calculator a rețelei electrice descrisă de problema pe contur particulară (I.1).

ON SOME FAMILIES OF NONLINEAR OPERATORS

BY

N. GRĂDINARU

This paper is a slight extension of some theorems given in [1], [3], [4] for nonlinear operators. Let X be a Banach space, $X_0 \subset X$ and

$$E_2^+ = \{\bar{x} = (\xi_1, \xi_2); \xi_1 \geq 0, \xi_2 \geq 0, (\xi_1, \xi_2) \neq (0, 0)\}.$$

A family of nonlinear operators $\{T(\bar{x}) = T(\xi_1, \xi_2), \bar{x} \in E_2^+ \text{ (where } T(x): X_0 \rightarrow X_0)\}$ is called a two-parameter semigroup of contractions on X_0 if:

- (i) $T(0, 0) = I, T(\bar{x}_1 + \bar{x}_2) = T(\bar{x}_1)T(\bar{x}_2)$ for $\forall \bar{x}_1, \bar{x}_2 \in E_2^+,$
- (ii) $\|T(\bar{x})x - T(\bar{y})y\| \leq \|x - y\|$ for $\forall \bar{x} \in E_2^+ \text{ and } \forall x, y \in X_0,$
- (iii) $\lim_{\|\bar{x}\| \rightarrow 0} T(\bar{x})x = x, \forall x \in X_0, (\|\bar{x}\| = \sqrt{\xi_1^2 + \xi_2^2}, \bar{x} = (\xi_1, \xi_2)).$

Let $A(\bar{x}) = \xi_1 A_1 + \xi_2 A_2$ be a dissipative operator in which A_1, A_2 are nonlinear multivalued operators in X . We shall use the notations: the domain of A is $D(A) = D(A_1) \cap D(A_2)$ the range is $R(A(\bar{x})) = \bigcup_{x \in D(A)} A(\bar{x})x,$

$$\|A(\bar{x})x\| = \inf_{x' \in A(\bar{x})x} \|x'\| \leq \xi_1 \|A_1 x\| + \xi_2 \|A_2 x\|,$$

and

$$J_{\alpha, \bar{x}} = (I - \alpha A(\bar{x}))^{-1} : R(I - \alpha A).$$

Theorem 1. *Let $A(\bar{x})$ be a dissipative operator in X . If*

$$(1) \quad R(I - \alpha A(\bar{x})) \supset \overline{D(A)}$$

for $\alpha > 0$, then there is a family $\{T(\bar{x}); \bar{x} \in E_2^+\}$ of contractions operators such that $T(\bar{x}) : \overline{D(A)} \rightarrow \overline{D(A)},$

$$(2) \quad T(\bar{x})x = \lim_{n \rightarrow \infty} (I - (\xi_1 A_1 + \xi_2 A_2)/n)^{-n} x, \quad x \in \overline{D(A)},$$

uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$,

$$(3) \quad \|T(t\bar{x})x - T(\tau\bar{x})x\| \leq 2 |t - \tau| \|A(\bar{x})x\|, \quad \forall t, \tau \geq 0, \quad \bar{x} \in E_2^+, \quad x \in \overline{D(A)}$$

and

$$(4) \quad \lim_{\|x\| \rightarrow 0} T(\bar{x})x = x, \quad \forall x \in \overline{D(A)}.$$

Proof. It follows, from [1] (lemma 1.2 for each $\bar{x}$), that the $\lim_{n \rightarrow \infty} J_{\frac{n}{n}, \bar{x}} x = T(\bar{x})x$ exists for $x \in \overline{D(A)}$ uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$.

Moreover, we deduce (ii) and (3) from the properties of $J_{\alpha, \bar{x}}$. Since $\lim_{n \rightarrow \infty} J_{\frac{n}{n}, \bar{x}} x = T(\bar{x})x$ is uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$, using the lemma 1.7.6. from [2], it follows $\lim_{\|x\| \rightarrow 0} T(\bar{x})x = x, x \in \overline{D(A)}$ (here $\bar{x} \leq \bar{x}'$ if $\xi_i \leq \xi'_i, i = 1, 2$).

Theorem 2. Let $\{A_n(\bar{x})\}$ be a sequence of dissipative operators in X . Suppose that

$$(5) \quad R(I - \alpha A_n(\bar{x})) \supset \overline{\text{co } D(A_n(\bar{x}))},$$

$$(6) \quad D \subset D(A_n(\bar{x})), \quad \forall n \in N \quad (\text{we denote } J_{\alpha, n, \bar{x}} = (I - \alpha A_n(\bar{x}))^{-1}, \quad \forall \alpha > 0),$$

$$(7) \quad \lim_{n \rightarrow \infty} J_{\alpha, n, \bar{x}} x (x \in \overline{\text{co } D})$$

exists in D for a number $\alpha_0 > 0$, uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$,

$$(8) \quad R(I - \alpha A(\bar{x})) \supset \overline{D(A)}$$

for $\alpha > 0$ (where $A(\bar{x})$ is given by lemma (2.2) [1] for each fixed $\bar{x}$).

If $\{T_n(\bar{x}); \bar{x} \in E_2^+\}$ (given by $A_n(\bar{x})$ and theorem 1) are semigroups of contractions, then the family $\{T(\bar{x}); \bar{x} \in E_2^+\}$ (given by A and theorem 1) is a semigroup of contractions and

$$\lim_{n \rightarrow \infty} T_n(\bar{x})x = T(\bar{x})x, \quad \forall x \in D(A), \quad \forall \bar{x} \in E_2^+.$$

uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$.

Proof. We denote $A_{\alpha, n, \bar{x}} = \alpha^{-1}(J_{\alpha, n, \bar{x}} - I)$ and $A_{\alpha, \bar{x}} = \alpha^{-1}(J_{\alpha, \bar{x}} - I)$ for $\alpha > 0$.

Since $\lim_{n \rightarrow \infty} J_{\alpha, n, \bar{x}} x = J_{\alpha, \bar{x}} x$, $x \in \overline{\text{co}D}$ for $\forall \alpha > 0$, uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$, (see [1]), it follows that $\lim_{n \rightarrow \infty} A_{\alpha, n, \bar{x}} x = A_{\alpha, \bar{x}} x$, $x \in \overline{\text{co}D}$, $\alpha > 0$, uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$ and that there is $M > 0$ such that $\|A_{\alpha_0, n, \bar{x}} x\| \leq M$, $\|A_{\alpha, \bar{x}} x\| \leq M$. Also, we have $A_{\alpha_0, \bar{x}} y \in A(\bar{x})J_{\alpha_0, \bar{x}} y$ and hence :

$$\|A_n(\bar{x})J_{\alpha_0, n, \bar{x}} y\| \leq \|A_{\alpha_0, n, \bar{x}} y\| \leq M.$$

From lemma 1.2 [1] we have

$$\begin{aligned} & \|J_{\frac{1}{n}, \bar{x}}^n x - J_{\frac{1}{m}, \bar{x}}^m x\| \leq \\ & \leq \left(\left| \frac{1}{m} - \frac{1}{n} \right| + \left| \frac{1}{m} - \frac{1}{n} \right| \right) \|A(\bar{x})x\| = 2 \left| \frac{1}{m} - \frac{1}{n} \right| \|A(\bar{x})x\| \leq \\ & \leq 2 \left| \frac{1}{m} - \frac{1}{n} \right| (\xi_1 \|A_1 x\| + \xi_2 \|A_2 x\|), \end{aligned}$$

and therefore :

$$\|T_n(\bar{x})x - J_{\frac{1}{k}, n, \bar{x}}^k x\| \leq \frac{2}{\sqrt{k}} \|A_n(\bar{x})x\|.$$

For fixed $y \in \overline{\text{co}D}$ and $\alpha_0 > 0$ we have

$$\begin{aligned} & \|T_n(\bar{x})J_{\alpha_0, \bar{x}} y - J_{\frac{1}{k}, n, \bar{x}}^k J_{\alpha_0, \bar{x}} y\| \leq \|T_n(\bar{x})J_{\alpha_0, \bar{x}} y - T_n(\bar{x})J_{\alpha_0, n, \bar{x}} y\| + \\ & + \|T_n(\bar{x})J_{\alpha_0, n, \bar{x}} y - J_{\frac{1}{k}, n, \bar{x}}^k J_{\alpha_0, n, \bar{x}} y\| + \|J_{\frac{1}{k}, n, \bar{x}}^k J_{\alpha_0, n, \bar{x}} y - \\ & - J_{\frac{1}{k}, n, \bar{x}}^k J_{\alpha_0, \bar{x}} y\| \leq 2 \|J_{\alpha_0, \bar{x}} y - J_{\alpha_0, n, \bar{x}} y\| + \|T_n(\bar{x})J_{\alpha_0, n, \bar{x}} y - \\ & - J_{\frac{1}{k}, n, \bar{x}}^k J_{\alpha_0, n, \bar{x}} y\| \leq 2 \|J_{\alpha_0, \bar{x}} y - J_{\alpha_0, n, \bar{x}} y\| + \frac{2}{\sqrt{k}} \|A_n(\bar{x})J_{\alpha_0, n, \bar{x}} y\| \leq \\ & \leq 2 \|J_{\alpha_0, \bar{x}} y - J_{\alpha_0, n, \bar{x}} y\| + \frac{2}{\sqrt{k}} M, \end{aligned}$$

where M is independent of k ,

$$\|J_{\frac{1}{k}, n, \bar{x}}^k J_{\alpha_0, \bar{x}} y - T(\bar{x})J_{\alpha_0, \bar{x}} y\| \leq \frac{2}{\sqrt{k}} \|A_n(\bar{x})J_{\alpha_0, \bar{x}} y\| \leq \frac{2}{\sqrt{k}} M,$$

and

$$\begin{aligned} & \| T_n(\bar{x}) J_{\alpha_0, \bar{x}} y - T(\bar{x}) J_{\alpha_0, \bar{n}} y \| \leq \| T_n(\bar{x}) J_{\alpha_0, \bar{x}} y - J_{\frac{1}{k}, n, \bar{x}} J_{\alpha_0, \bar{x}} y \| + \\ & + \| J_{\frac{1}{k}, n, \bar{x}} J_{\alpha_0, \bar{x}} y - J_{\frac{1}{k}, \bar{x}} J_{\alpha_0, \bar{x}} y \| + \| J_{\frac{1}{k}, \bar{x}} J_{\alpha_0, \bar{x}} y - T(\bar{x}) J_{\alpha_0, \bar{x}} y \|. \end{aligned}$$

Finally

$$\begin{aligned} & \| T_n(\bar{x}) J_{\alpha_0, \bar{x}} y - T(\bar{x}) J_{\alpha_0, \bar{x}} y \| \leq 2 \| J_{\alpha_0, \bar{x}} y - J_{\alpha_0, n, \bar{x}} y \| + \\ & + \| J_{\frac{1}{k}, n, \bar{x}} J_{\alpha_0, \bar{x}} y - J_{\frac{1}{k}, \bar{x}} J_{\alpha_0, \bar{x}} y \| + \frac{4}{\sqrt{k}} M. \end{aligned}$$

Therefore, $\limsup_{n \rightarrow \infty} \| T_n(\bar{x}) J_{\alpha_0, \bar{x}} y - T(\bar{x}) J_{\alpha_0, \bar{x}} y \| \leq \frac{4}{\sqrt{k}} M$ and for $k \rightarrow \infty$ we have $\lim_{n \rightarrow \infty} T_n(\bar{x}) J_{\alpha_0, \bar{x}} y = T(\bar{x}) J_{\alpha_0, \bar{x}} y$, uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$. But $R(J_{\alpha_0, \bar{x}}) = D(A) \subset D$ implies $\forall x \in D(A)$ there is $y \in \overline{\text{co}D}$ such that $x = J_{\alpha_0, \bar{x}} y$, hence $\lim_{n \rightarrow \infty} T_n(\bar{x}) x = T(\bar{x}) x$, $\forall x \in D(A)$ and because $T_n(\bar{x})$ and $T(\bar{x})$ are contractions it results that $\lim_{n \rightarrow \infty} T_n(\bar{x}) x = T(\bar{x}) x$ for $x \in \overline{D(A)}$ uniformly on every bounded subset of $E_2^+ \cup \{(0, 0)\}$.

The semigroup property of $T(\bar{x})$ results from :

$$\begin{aligned} & \| T(\bar{x}_1 + \bar{x}_2) y - T(\bar{x}_1) T(\bar{x}_2) y \| \leq \| T(\bar{x}_1 + \bar{x}_2) y - T_n(\bar{x}_1 + \bar{x}_2) y \| + \\ & + \| T_n(\bar{x}_1) T_n(\bar{x}_2) y - T_n(\bar{x}_1) T(\bar{x}_2) y \| + \| T_n(\bar{x}_1) T(\bar{x}_2) y - T(\bar{x}_1) T(\bar{x}_2) y \| \leq \\ & \leq \| T(\bar{x}_1 + \bar{x}_2) y - T_n(\bar{x}_1 + \bar{x}_2) y \| + \| T_n(\bar{x}_2) y - T(\bar{x}_2) y \| + \\ & + \| T_n(\bar{x}_1) T(\bar{x}_2) y - T(\bar{x}_1) T(\bar{x}_2) y \| \rightarrow 0 \text{ for } n \rightarrow \infty. \end{aligned}$$

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ASUPRA UNOR FAMILII DE OPERATORI NELINIARI

(Rezumat)

Se consideră unele extensiuni ale unor rezultate din [1], [3], [4] pentru semigrupuri de operatori neliniari.

D.C. 517.9

ON A NON-LINEAR RECURSION RELATION

BY

S. G. GHURYE and WILLIAM S. GUSTIN

0. Introduction

This note arises out of the following problem posed by Thomas R. Crimmins (oral communication): Does there exist a positive sequence $\{x_n, n=0, 1, 2, \dots\}$ satisfying the equation $x_{n-1}x_{n+1} = x_n - x_0$, ($n = 1, 2, \dots$)? This problem was encountered by Crimmins in the study of non-normal operators in Hilbert space, specifically in seeking to construct examples of operators T such that $T^*T - TT^*$ is a projection of rank 1. The problem, in addition to being non-linear, has the further disadvantage of not being analogous to a problem in differential equations. Consequently, there do not seem to be any standard, sophisticated methods of solution. Several empirical attempts at solution led to failure, suggesting that there are few, if any, positive sequences of the type desired.

Motivated by the problem of Crimmins, we consider the recursion relation

$$(0.1) \quad x_{n-1}x_{n+1} = x_n - \alpha,$$

where α is an arbitrary real number, and investigate the following question:

Does there exist a sequence of positive numbers $x_0, x_1, x_2, \dots$, which satisfies (0.1) for all positive n , with the initial condition $x_0 = \alpha$? If so, what is it like?

We first construct a solution of this problem; specifically, we define for each $x_0 \in (0, 1/4)$ an x_1 such that the resulting sequence is positive. We then show that this is a rather special solution in as much as it is the only monotone solution, and that no smaller value of x_1 gives a positive solution. It is also suggested that the solution obtained is rather

special, and perhaps unique. We are unable to prove uniqueness by the simple methods of this paper; this result will be derived in a subsequent paper by a different approach.

1. Existence and Construction of Positive Solutions

In this section, we shall be concerned with the existence and nature of positive solutions. To begin with, we notice that if we did not have the initial condition $x_0 = \alpha$, a trivial solution would be provided by $x_n = = \xi$ where ξ is a zero of the polynomial

$$(1.1) \quad p(x) = x^2 - x + \alpha.$$

We also notice that if a sequence satisfying (0.1) converges as $n \rightarrow \infty$, then the limit is a zero of the polynomial p . It is thus clear that this polynomial and its zeroes will play an important role in our discussion. There are three situations to consider:

- (i) $\alpha \leq 0$, in which case, the zeroes of p are real, one being ≤ 0 .
- (ii) $0 < \alpha \leq 1/4$, in which case, the zeroes are real and positive;
- (iii) $1/4 < \alpha$, in which case, the zeroes have non-vanishing imaginary parts.

In Case (i), there is no positive sequence satisfying (0.1) with the initial condition $x_0 = \alpha$; on the other hand, if there were no initial condition, every positive x_0 and x_1 would generate a positive sequence satisfying (0.1). Hence, this case is not of interest.

In Case (iii), $p(x) \geq p_0 > 0$ for all real x ; this implies the non-existence of positive solutions, with or without the initial condition $x_0 = \alpha$. The following is a proof of this assertion: Let

$$(1.2) \quad r_n = \frac{x_n}{x_{n-1}},$$

and observe that (0.1) can be rewritten as

$$(1.3) \quad r_n - r_{n+1} = \frac{p(x_n)}{x_{n-1}x_n}.$$

Hence, $x_{n-1}x_n > 0 \Rightarrow r_n > r_{n+1}$. Consequently, if $r_m \leq 1$ for some m , then $r_n \leq \leq r < 1$ for $n > m$, and $\{x_n, n > m\}$ is dominated by a geometric progression (with ratio r) which converges to 0. Hence, $x_n < \alpha_n$ for some n , and $x_{n+1} < 0$. Therefore, we must have $r_n > 1$ for $n = 1, 2, \dots$, if the x_n are positive for all positive n . In that case, $\{x_n, n = 1, 2, \dots\}$ is a monotone increasing sequence; if it were bounded, it would converge to a finite limit x and we would have $p(x) = 0$, which is impossible. Hence, $\{x_n, n = 1, 2, \dots\}$ must be an unbounded increasing sequence if it is to be positive. Therefore, eventually the l. h. s. of (0.1) exceeds x_{n+1} whereas the

r. h. s. is less than x_{n+1} , so that this possibility is also ruled out. This shows that in Case (iii), there are no positive solutions.

The remaining Case (ii), $0 < \alpha \leq 1/4$, is of considerable interest. Since $\alpha = 1/4$ is a limiting case and can be dealt with by usual limit arguments, we shall henceforward assume $0 < \alpha < 1/4$.

Let ξ and ξ' be the zeroes of p , with $\xi < \xi'$; note that $\xi\xi' = a$, $\xi + \xi' = 1$ and $\alpha < \xi < 1/2 < \xi' < 1$.

Now, empirically constructed sequences satisfying (0.1) seem to be of two kinds: (i) those that begin to decrease before reaching ξ , eventually becoming negative, and (ii) those that increase monotonically beyond ξ and then decrease below 0. However, x_n is a rational continuous function of (x_0, x_1) on any set on which $x_2, x_3, \dots, x_{n-1}$ are continuous and positive. This suggests the possibility that there exists a choice of (x_0, x_1) such that the solution of

$$(1.4) \quad x_{n-1}x_{n+1} = x_n - x_0, \quad (n = 1, 2, \dots),$$

is a monotone increasing sequence bounded above by ξ (and hence convergent, with limit $= \xi$).

Pursuing this possibility, we set out to find $\{x_n, n = 1, 2, \dots\}$ such that

$$(1.5) \quad 0 < x_n < x_{n+1} = \frac{x_n - \alpha}{x_{n-1}} < \xi, \quad (n = 1, 2, \dots).$$

Using the fact that $\alpha = \xi - \xi^2$, these relations can be re-written as

$$(1.6) \quad -\xi^2 < x_n x_{n-1} - \xi^2 < x_{n+1} x_{n-1} - \xi^2 = x_n - \xi < \xi(x_{n-1} - \xi) < 0.$$

Making the substitution $w_n = (\xi - x_n)^{-1}$, we have

$$(1.7) \quad \xi w_n < w_{n-1} < w_n,$$

$$(1.8) \quad 0 < \xi^{-1} w_{n-1} - w_n < w_{n-1} - \xi^{-1},$$

$$(1.9) \quad w_n(w_{n+1} + w_{n-1}) = \xi^{-1}(w_n + w_{n-1}w_{n+1}).$$

An outburst of optimism at this stage would suggest replacing (1.8) by

$$(1.10) \quad \xi^{-1} w_{n-1} - w_n = w_{n-2} - \xi^{-1}, \quad (n = 2, 3, \dots),$$

since, by virtue of (1.7), the r. h. s. in (1.10) is less than the r. h. s. in (1.8).

The general solution of (1.10) is

$$(1.11) \quad w_n = A\rho^n + B\rho^{-n} + C, \quad (n = 0, 1, 2, \dots),$$

where A, B are arbitrary constants and ρ is the larger root of $\xi x^2 - x + \xi = 0$. Note that since $\xi < 1/2$, the roots (ρ, ρ^{-1}) are real and positive, with $\rho > 1$. If we can find A, B, C so as to satisfy (1.7), (1.9) and (1.11), then we shall have constructed a sequence $\{x_n\}$ satisfying (1.5). Now,

for any positive A , w_n as defined by (1.11) satisfies (1.7) for sufficiently large n ; therefore, the problem reduces essentially to verifying whether there exist A , B , C such that (1.9) and (1.11) hold. For this purpose, we shall rewrite (1.9) as

$$(1.12) \quad w_n(w_{n+1} - \xi^{-1}w_n + w_{n-1}) = \xi^{-1}(w_n - w_n^2 + w_{n-1}w_{n+1}).$$

Then (1.9) and (1.11) are together equivalent to (1.11) and

$$(1.13) \quad \xi^{-1}(w_n^2 - w_{n-1}w_{n+1}) = \{C(\xi^{-1} - 2) + \xi^{-1}\}w_n.$$

On substituting from (1.11) into (1.13) and remembering that $\rho + \rho^{-1} = \xi^{-1}$, several terms drop out, add by a stroke of good luck, we are left with

$$(1.14) \quad \begin{aligned} C(2 - \xi^{-1})(A\rho^n + B\rho^{-n}) - AB(\xi^{-2} - 4) = \\ = \{C(1 - 2\xi) + 1\}\{A\rho^n + B\rho^{-n} + C\}, \end{aligned}$$

which is satisfied for all n if (and only if)

$$(1.15) \quad \begin{cases} C(1 + \xi)(1 - 2\xi) = \xi, \\ AB(1 + 2\xi) = \xi C^2. \end{cases}$$

Further, in order to satisfy (1.7), it is necessary and sufficient that $A\rho^{2n-1} > B$, so that both (1.7) and (1.9) are satisfied for $n = 1, 2, \dots$, if and only if

$$(1.16) \quad A\rho > B \quad \text{and} \quad A + B + C > \xi^{-1}.$$

Consequently, we have a continuum of positive sequences satisfying (0.1).

Finally, going back to the original problem of Crimmins of (0.1) with the initial condition $x_0 = \alpha$, we have another condition on A and B , viz.,

$$(1.17) \quad A + B + C = w_0 = (\xi - x_0)^{-1} = (\xi - \alpha)^{-1} = \xi^{-2}.$$

It turns out there exists one (and only one) solution of (1.15) and (1.17) which satisfies (1.16); this is readily seen from the fact that, since $\xi < 1/2$, (1.17) implies the second inequality in (1.16), and that (1.15) and (1.17) imply

$$(1.18) \quad \begin{cases} ABC^{-2} = (\xi^{-1} + 2)^{-1} < \frac{1}{4}, \\ (A+B)(-C)^{-1} = 1 + \xi^{-1}(\xi^{-1} + 1)(\xi^{-1} - 2) > 1. \end{cases}$$

There are two solutions of (1.18), of which the one with $0 < B < A$ satisfies (1.16) as well as (1.15) and (1.17), and it is the only one that does.

Thus, by a combination of optimistic unorthodoxy and good fortune, we have succeeded in constructing a sequence

$$(1.19) \quad x_n = \xi - (A\rho^n + B\rho^{-n} + C)^{-1}$$

which satisfies (1.15).

Actually, we found that the work is a little simplified if (1.19) is written in the form

$$(1.20) \quad x_n = \xi - \frac{1 + \xi}{y_n},$$

where

$$(1.21) \quad \begin{cases} y_n = (a\rho^n + b\rho^{-n} - 1) \frac{\xi}{1 - 2\xi}, \\ ab = \frac{\xi}{1 + 2\xi}. \end{cases}$$

The initial condition, $x_0 = \alpha$, now becomes

$$(1.22) \quad \xi^3(a + b - 1) = (1 + \xi)(1 - 2\xi).$$

Remembering that we want the larger of the two possible values of a , we find that the unique solution of the form defined by (1.20), (1.21) and (1.22) is given by the initial values

$$(1.23) \quad y_0 = \xi^{-2} + \xi^{-1}, \quad y_1 = \xi^{-3} + \xi^{-2} - \xi^{-1},$$

or

$$(1.24) \quad x_0 = \alpha = \xi - \xi^2, \quad x_1 = \frac{\xi + \xi^2 - 2\xi^3 - \xi^4}{1 + \xi - \xi^2}.$$

Having now found a solution to the problem of Crimmins, we naturally wonder how many such solutions exist. From the empirical evidence that these solutions are hard to find, we surmise that they are rare. The following theorem shows that the solution we have found is indeed a rather special one.

Theorem. Let $\{x_n(\alpha, \beta), n = 0, 1, 2, \dots\}$ be defined by $x_0(\alpha, \beta) = \alpha$, $x_1(\alpha, \beta) = \beta$, and

$$(1.25) \quad x_{n-1}(\alpha, \beta)x_{n+1}(\alpha, \beta) = x_n(\alpha, \beta) - \alpha, \quad (n = 1, 2, \dots).$$

Let $\hat{\beta} = (\xi + \xi^2 - 2\xi^3 - \xi^4)/(1 + \xi - \xi^2)$, where ξ is the smaller root of $p(x) \equiv x^2 - x + \alpha = 0$. Then for each $\alpha \in (0, 1/4)$,

- (i) $\{\hat{x}_n = x_n(\alpha, \beta), n = 0, 1, 2, \dots\}$ is the only monotone solution of (1.25);
- (ii) there does not exist a $\beta < \hat{\beta}$ which yields a positive solution;
- (iii) there does not exist a $\beta' > \hat{\beta}$ such that every $\beta \in [\hat{\beta}, \beta']$ generates a positive solution of (1.25).

Proof. We have already seen that $\{\hat{x}_n\}$ is a positive solution and that it is a strictly increasing sequence converging to ξ . In order to study the behaviour of sequences corresponding to values of β other than $\hat{\beta}$ for any α , let

$$(1.26) \quad \begin{cases} r_n = \frac{x_n}{x_{n-1}} \\ \hat{r}_n = \frac{\hat{x}_n}{\hat{x}_{n-1}} \end{cases}, \quad (n = 1, 2, \dots),$$

so that (1.25) can be rewritten as

$$(1.27) \quad r_{n+1} = r_n - \frac{p(x_n)}{x_{n-1}x_n}, \quad (n = 1, 2, \dots).$$

We shall use repeatedly the fact that $p(x)$ is positive and decreasing for $x < \xi$, negative for $x \in (\xi, \xi')$, and positive and increasing for $x > \xi'$.

Case A. Let $\beta \leq 0$. If $\beta = 0$, then $x_2 = -1$ and there does not exist an x_3 satisfying (1.25). On the other hand, if $\beta < 0$, then $x_2 < 0$, and hence $x_3 > 0$; consequently, the sequence $\{x_n\}$ is neither monotone nor positive.

Case B. Let $0 < \beta < \hat{\beta}$. Then $x_1(\alpha, \beta) = \beta < \hat{x}_1$ and $r_1 < \hat{r}_1$. Now suppose

$$(1.28) \quad 0 < x_n < \hat{x}_n \quad \text{and} \quad r_n < \hat{r}_n, \quad (n = 1, 2, \dots, k).$$

Then $p(x_n) > p(\hat{x}_n)$, $(n = 1, 2, \dots, k)$, and hence

$$(1.29) \quad r_{k+1} = r_k - \frac{p(x_k)}{x_{k-1}x_k} < \hat{r}_k - \frac{p(\hat{x}_k)}{\hat{x}_{k-1}\hat{x}_k} = \hat{r}_{k+1}$$

and

$$\begin{cases} x_n = x_1 \frac{x_2 \dots x_n}{x_1 \dots x_{n-1}} = \beta r_2 \dots r_n, \\ \hat{x}_n = \hat{\beta} \hat{r}_2 \dots \hat{r}_n, \end{cases} \quad (n = 1, 2, \dots),$$

so that

$$(1.30) \quad x_n = \frac{x_n}{\hat{x}_n} \hat{x}_n = \frac{\beta r_2 \dots r_n}{\hat{\beta} \hat{r}_2 \dots \hat{r}_n} \hat{x}_n < \left(\frac{\beta}{\hat{\beta}} \right) \hat{x}_n, \quad (n = 1, 2, \dots, k+1).$$

Also, $p(x_n) > 0$, $(n = 1, 2, \dots, k)$, so that

$$(1.31) \quad r_1 > r_2 > \dots > r_{k+1}.$$

If $x_{k+1} > 0$, then the inequalities (1.28) hold for $n = k+1$. Hence, either $x_{m+1} \leq 0$ for some m or (1.28) holds for all k ; it is easy to see that the latter is impossible.

For suppose that (1.28) holds for all k . Then either $r_n > 1$ for all n or $r_l \leq 1$ for some l . In the first case, $\{x_n\}$ would be an increasing sequence bounded above by $\xi\beta/\hat{\beta}$, which is impossible since $\lim x_n$ has to be a zero of $p(x)$. In the second case, we have $r_{l+i} < 1$, ($i = 1, 2, \dots$), so that $x_{l+1} = x_l r_{l+1} \dots r_{l+i-1} < x_l (r_{l+1})^{i-1}$; hence, for some m , $x_{m-1} < \alpha$, so that $x_m < 0$.

Thus there is no $\beta < \hat{\beta}$ such that $\{x_n(\alpha, \beta), n = 0, 1, 2, \dots\}$ is a positive or a monotone sequence. We have now proved Assertion (ii) of the theorem and are on the way to proving Assertion (i).

Case C. Let $\beta > \hat{\beta}$. Then $x_1(\alpha, \beta) > \hat{x}_1$ and $\hat{r}_1 > r_1$. Now, suppose

$$(1.32) \quad \hat{x}_n < x_n < \xi, \quad \hat{r}_n < r_n, \quad (n = 1, 2, \dots, k).$$

By an argument similar to the one used in Case B, we have

$$(1.33) \quad \hat{r}_{k+1} < r_{k+1} \quad \text{and} \quad x_n > \left(\frac{\beta}{\hat{\beta}}\right) \hat{x}_n, \quad (n = 1, 2, \dots, k+1).$$

Since $\hat{x}_n \rightarrow \xi$, (1.13) implies that $x_n > \xi$ for some n ; but from (1.27) and the observation following it, we see that $x_n \in (\xi, \xi')$ implies $r_{n+1} > r_n$. Hence x_n continues to increase beyond ξ' . Now there are three possibilities: $\{x_n\}$ is either an increasing sequence bounded above by 1, or an increasing sequence with $x_m \geq 1$ for some m , or a non-monotone sequence. The first alternative is ruled out by the fact that it would imply the existence of $\lim x_n$ which would be a zero of $p(x)$ although it is larger than ξ' . On the other hand, if $x_m > 1$, then (1.25) gives us

$$(1.34) \quad r_{m+1} = \frac{1}{x_m} - \frac{\alpha}{x_m x_{m-1}} < 1.$$

Consequently, $\beta > \hat{\beta}$ implies that $\{x_n\}$ is not a monotone sequence, thus completing the proof of Assertion (i).

In fact, we have seen that if $\beta > \hat{\beta}$, then there exists an m such that $\beta = x_1 < x_2 < \dots < x_m$ but $x_m \geq x_{m+1}$, where $x_m > \xi'$. From (1.27) and the observation following it, we see that $x_u > \xi'$ implies $r_{n+1} < r_n$. By an argument like used in the discussion of Case B, we see that, for $n \geq m$, $\{x_n\}$ is dominated by a decreasing geometric progression; hence, there exists an m' such that $x_m > x_{m+1} > \dots > x_{m'} \geq \xi' > x_{m'+1}$. These considerations help us prove Assertion (iii).

Suppose Assertion (iii) is not true, so that there exists a $\beta' > \hat{\beta}$ such that every $\beta \in [\hat{\beta}, \beta']$ generates a positive solution of (1.25). Let $x'_n = x_n(\alpha, \beta')$, ($n = 1, 2, \dots$). Then there exists an m such that $x'_m \geq x'_{m+1}$. From (1.25), we see immediately that every $x_n(\alpha, \beta)$ is, for fixed α , a con-

tinuous function of $\beta \in [\hat{\beta}, \beta']$. Hence, if $x'_m > x'_{m+1}$, we note that $x_m(\alpha, \beta) - x_{m+1}(\alpha, \beta)$ is a continuous function which is positive at $\beta = \beta'$ and negative at $\beta = \hat{\beta}$; so it vanishes at some point in $[\hat{\beta}, \beta']$. Thus the assumption made at the beginning of the paragraph implies the existence of a $\beta_0 \in [\hat{\beta}, \beta']$ such that $x_m(\alpha, \beta_0) = x_{m+1}(\alpha, \beta_0)$. From (1.25) we see immediately that this implies $x_{m-1}(\alpha, \beta_0) = x_{m+2}(\alpha, \beta_0)$, and hence successively, $x_{m-2}(\alpha, \beta_0) = x_{m+3}(\alpha, \beta_0), \dots, \alpha = x_0(\alpha, \beta_0) = x_{2m+1}(\alpha, \beta_0)$; so that we have finally $x_{2m+2}(\alpha, \beta_0) = 0$, contradicting the assumption that Assertion (iii) is not true.

2. Concluding Remarks

The discussion of the previous section indicates that the solution, which has been obtained is rather special: no smaller value of x_1 works and if there are larger values which generate positive sequences, they are scarce. This situation suggests the possibility that the solution we have found is the unique solution, that is to say, to each $x_0 \in (0, 1/4)$, there corresponds only one x_1 which generates a positive sequence. However, the simple approach adopted here, which has succeeded beyond expectation, is inadequate to establish the uniqueness of the solution. The main reason for this inadequacy is, of course, that in order to be able to predict the behaviour of the entire sequence from a few terms, we must achieve some global understanding. At the same time, it is true that every sequence defined by (0.1) is completely determined by a pair of consecutive elements. To be more precise, if $(x_0, x_1, \dots, x_m)$ and $(x'_0, x'_1, \dots, x'_m)$ are non-zero numbers such that $x_{n-1}x_{n+1} = x_n - \alpha$ and $x'_{n-1}x'_{n+1} = x'_n - \alpha$ for $n = 1, 2, \dots, m-1$, and if $x_k = x'_k, x_{k+1} = x'_{k+1}$ for some $k \in \{0, 1, \dots, m-1\}$, then $x_i = x'_i, i = 0, 1, \dots, m$. This property suggests the idea of looking at solutions in terms of pairs of consecutive elements, which will be done in a subsequent paper, devoted to a functional equation associated with a non-linear recursion. We are, by these means, able to establish the uniqueness of the solution to our problem.

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O RELATIE DE RECURENȚĂ NELINIARĂ

(Rezumat)

Se studiază problema existenței unui șir de numere pozitive $\{x_n, n = 0, 1, 2, \dots\}$ astfel încît $x_{n-1}x_{n+1} = x_n - x_0, n = 1, 2, \dots$. Încercările empirice de obținere a soluției nu au dat rezultate. Se arată că nu există soluție pentru $x_0 > 1/4$; dar pentru fiecare $x_0 \in (0, 1/4]$ corespunde un număr $\hat{x}_1$ astfel încît șirul generat de $x_1 = \hat{x}_1$ este pozitiv; acest $\hat{x}_1$ este cel mai mic x_1 care generează un șir pozitiv. Valoarea critică $\hat{x}_1$ este $(\xi + \xi^2 - 2\xi^3 - \xi^4)/(1 + \xi - \xi^2)$, în care ξ este cea mai mică rădăcină a lui $\xi^2 - \xi + x_0$.

THE SECOND (NON-PERIODIC) SOLUTION OF THE WHITTAKER-HILL EQUATION FOR SMALL PARAMETERS

BY

KATHLEEN M. URWIN and S. Y. CHONG

1. Introduction

The Whittaker-Hill equation

$$(1.1) \quad \frac{d^2\Phi}{d\theta^2} + \left[\eta + \frac{1}{8} \omega^2 + \rho \omega \cos 2\theta - \frac{1}{8} \omega^2 \cos 4\theta \right] \Phi = 0$$

arises when the Helmholtz equation is separated in general paraboloidal coordinates. The theory of this equation is fully discussed by Urwin and Arscott [5]; we shall quote here only those results essential to the work of this paper.

In equation (1.1) ω may be taken as known a priori, whilst η and ρ are disposable parameters. Paraboloidal wave functions are basically periodic solutions of (1.1), that is they have period or anti-period π and are even or odd in θ . There are thus four types of function, which we denote as follows:

- $gc_{2n}(\theta, \omega, \rho)$; period π , even,
- $gc_{2n+1}(\theta, \omega, \rho)$; anti-period π , even,
- $gs_{2n+1}(\theta, \omega, \rho)$; anti-period π , odd,
- $gs_{2n+2}(\theta, \omega, \rho)$; period π , odd, ($h = 0, 1, 2, 3, \dots$).

For given ω, ρ these solutions exist only if η satisfies a transcendental equation; for each solution $\eta = \eta(\omega, \rho)$ takes one of an infinite set of values which is a solution for η of the appropriate transcendental equation. Corresponding to $gc_n(\theta, \omega, \rho)$, $gs_n(\theta, \omega, \rho)$ we denote these characteristic values of η by $\eta_{c,n}(\omega, \rho)$, $\eta_{s,n}(\omega, \rho)$ respectively.

For general values of ω , ρ explicit expressions for the paraboloidal wave functions cannot be given, since they depend in a complicated way on infinite continued fractions involving η , ρ and ω : (see U r w i n and A r s c o t t [5, pp. 30–36]). For small ω , ρ however, perturbation solutions were obtained by U r w i n [4]. Equation (1.1) reduces, as $\omega \rightarrow 0$, to

$$(1.2) \quad \frac{d^2\Phi}{d\theta^2} + \eta\Phi = 0;$$

the paraboloidal wave functions $gc_n(\theta, \omega, \rho)$, $gs_n(\theta, \omega, \rho)$ reduce to $\cos n\theta$ $\sin n\theta$ respectively (except for $gc_0(\theta, \omega, \rho)$ which reduces to $1/\sqrt{2}$), and $\eta_{c,n}$, $\eta_{s,n}$ reduce to n^2 .

To obtain the perturbation form of $gc_n(\theta, \omega, \rho)$, for example, we assume

$$(1.3) \quad gc_n(\theta, \omega, \rho) \equiv \Phi \equiv \cos n\theta + \omega f_1(\theta) + \omega^2 f_2(\theta) + \omega^3 f_3(\theta) + \dots$$

$$(1.4) \quad \eta_{c,n} = n^2 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \dots,$$

where the coefficients $a_1, a_2, \dots$ are functions of ρ and the functions $f_1(\theta)$, $f_2(\theta)$, ... also involve ρ . For the perturbation form it is convenient to let gc_0 reduce to 1 as $\omega \rightarrow 0$. Using (1.3) and (1.4) in (1.1) and equating the coefficient of each power of ω to zero, we obtain a set of second order linear differential equations with constant coefficients for the functions $f_r(\theta)$. We normalise the solutions by making the coefficient of $\cos n\theta$ unity for all ω . In addition we ensure that the solutions is even and of appropriate period: this leads to the determination of the coefficients $a_1, a_2, \dots$. The detailed solutions are given by U r w i n [4]; here we quote one example, namely

$$(1.5) \quad \begin{aligned} gc_1(\theta, \omega, \rho) = & \cos \theta + \omega A_{1,3}^{(1)} \cos 3\theta + \omega^2 [A_{2,3}^{(1)} \cos 3\theta + A_{2,5}^{(1)} \cos 5\theta] + \\ & + \omega^3 [A_{3,3}^{(1)} \cos 3\theta + A_{3,5}^{(1)} \cos 5\theta + A_{3,7}^{(1)} \cos 7\theta] + \\ & + \omega^4 [A_{4,3}^{(1)} \cos 3\theta + A_{4,5}^{(1)} \cos 5\theta + A_{4,7}^{(1)} \cos 7\theta + A_{4,9}^{(1)} \cos 9\theta] + O(\omega^5), \end{aligned}$$

$$(1.6) \quad \eta_{c,1} = 1 - \frac{\rho}{2}\omega - \frac{(\rho^2 + 4)}{2^5}\omega^2 + \frac{\rho(\rho^2 + 4)}{2^9}\omega^3 - \frac{(\rho^2 + 4)^2}{3 \cdot 2^{13}}\omega^4 + O(\omega^5),$$

where

$$\begin{aligned} A_{1,3}^{(1)} &= \frac{\rho}{2^4}, & A_{2,3}^{(1)} &= -\frac{\rho^2 + 2}{2^8}, & A_{2,5}^{(1)} &= \frac{\rho^2 - 2}{3 \cdot 2^8}, \\ A_{3,3}^{(1)} &= \frac{\rho(\rho^2 + 4)}{3 \cdot 2^{12}}, & A_{3,5}^{(1)} &= -\frac{\rho(\rho^2 + 1)}{3^2 \cdot 2^{10}}, & A_{3,7}^{(1)} &= \frac{\rho(\rho^2 - 8)}{3^2 \cdot 2^{13}}, \end{aligned}$$

$$A_{4,3}^{(1)} = \frac{\rho^2(11\rho^2 + 38)}{3^2 \cdot 2^{16}}, \quad A_{4,5}^{(1)} = \frac{\rho^2(\rho^2 + 4)}{3 \cdot 2^{17}},$$

$$A_{4,7}^{(1)} = -\frac{\rho^4 - 4\rho^2 - 8}{3 \cdot 2^{14}}, \quad A_{4,9}^{(1)} = \frac{\rho^4 - 20\rho^2 + 24}{5 \cdot 3^2 \cdot 2^{18}}.$$

The perturbation solutions were originally denoted by Urwin [4] by $wh_{c,n}$, $wh_{s,n}$ to distinguish them from the solutions for general ω , ρ which are differently normalised (see Urwin and Arscott [5, p. 34]), but this distinction is unnecessary here.

2. The Second Solution

It is well known that when the first solution of (1.1) is basically-periodic the second solution is not periodic at all. This result was established by Ince [2] for Mathieu's equation and carries over to the Whittaker-Hill equation. Since the Whittaker-Hill equation always has one even and one odd solution we may take the second solution odd when the first is even, and even when the first is odd.

Denoting the second solutions corresponding to $gc_n(\theta, \omega, \rho)$ and $gs_n(\theta, \omega, \rho)$ by $kc_n(\theta, \omega, \rho)$, $ls_n(\theta, \omega, \rho)$ respectively, we have

$$(2.1) \quad kc_n(\theta, \omega, \rho) = C_n(\omega, \rho) \theta gc_n(\theta, \omega, \rho) + k_n(\theta, \omega, \rho),$$

$$(2.2) \quad ls_n(\theta, \omega, \rho) = S_n(\omega, \rho) \theta gs_n(\theta, \omega, \rho) + l_n(\theta, \omega, \rho),$$

where $k_n(\theta, \omega, \rho)$ is odd in θ , of the same period as gc_n , and $l_n(\theta, \omega, \rho)$ is even in θ , with the same period as gs_n .

Since the even and odd basically-periodic solutions of (1.2) are $\cos n\theta$, $\sin n\theta$, we see that as $\omega \rightarrow 0$, for $n \neq 0$, $k_n(\theta, \omega, \rho) \rightarrow \sin n\theta$, $l_n(\theta, \omega, \rho) \rightarrow \cos n\theta$,

$$C_n(\omega, \rho) \rightarrow 0, \quad S_n(\omega, \rho) \rightarrow 0.$$

For $n = 0$, $k_0(\theta, \omega, \rho) \rightarrow 0$ as $\omega \rightarrow 0$, and $C_0(\omega, \rho) \equiv 1$.

Substituting from (2.1) into (1.1) we find that $k_n(\theta, \omega, \rho)$ satisfies the equation

$$(2.3) \quad k_n''(\theta) + 2C_n gc_n'(\theta) + \left[\eta + \frac{1}{8} \omega^2 + \rho \omega \cos 2\theta - \frac{1}{8} \omega^2 \cos 4\theta \right] k_n(\theta) = 0.$$

To obtain the perturbation form of $k_n(\theta, \omega, \rho)$ we now assume

$$(2.4) \quad k_n(\theta, \omega, \rho) = \sin n\theta + \sum_{r=1}^{\infty} f_r(\theta) \omega^r,$$

$$(2.5) \quad C_n(\omega, \rho) = \sum_{r=1}^{\infty} \alpha_r \omega^r, \quad (n \neq 0),$$

$$(2.6) \quad C_0(\omega, \rho) = 1.$$

Using these forms and the perturbation form of $gc_n(\theta, \omega, \rho)$ in (2.3), we obtain a set of second-order linear differential equations with constant coefficients from which to determine successively $f_1(\theta)$, $f_2(\theta)$, By ensuring that the coefficient of $\sin n\theta$ is unity for all ω , and that $f_i(\theta)$ is odd with the same period as gc_n , the coefficients $\alpha_1, \alpha_2, \dots$ are successively found. In theory the solution may be determined to any desired power of ω , but the working rapidly becomes complicated.

3. The Perturbation Solutions

The perturbation forms of $k_n(\theta, \omega, \rho)$, $l_n(\theta, \omega, \rho)$ have been determined up to $O(\omega^4)$ for $n = 0, 1, 2, \dots, 6$. The forms are lengthy and we quote in detail only those for $n = 0, 1, 2$, with the corresponding constants C_n, S_n . For convenience we denote the coefficient of $\omega^m \sin r\theta$ in $k_n(\theta, \omega, \rho)$ by $D_{m,r}^{(n)}$, and that of $\omega^m \cos r\theta$ in $l_n(\theta, \omega, \rho)$ by $E_{m,r}^{(n)}$.

$n = 0$

$$\begin{aligned} k_0(\theta, \omega, \rho) = & \omega D_{1,2}^{(0)} \sin 2\theta + \omega^2 D_{2,4}^{(0)} \sin 4\theta + \\ & + \omega^3 [D_{3,2}^{(0)} \sin 2\theta + D_{3,6}^{(0)} \sin 6\theta] + \\ & + \omega^4 [D_{4,4}^{(0)} \sin 4\theta + D_{4,8}^{(0)} \sin 8\theta] + O(\omega^5), \end{aligned}$$

where

$$\begin{aligned} D_{1,2}^{(0)} = -\frac{\rho}{2^2}, \quad D_{2,4}^{(0)} = \frac{1 - 3\rho^2}{2^{10}}, \quad D_{3,2}^{(0)} = \frac{3\rho(9\rho^2 + 1)}{2^{11}}, \\ D_{3,6}^{(0)} = -\frac{\rho(11\rho^2 - 14)}{3^2 \cdot 2^{11}}, \quad D_{4,4}^{(0)} = \frac{\rho^2(20\rho^2 + 8)}{3^3 \cdot 2^{12}}, \\ D_{4,8}^{(0)} = -\frac{25\rho^4 - 238\rho^2 + 81}{3^3 \cdot 2^{19}}, \end{aligned}$$

and

$$C_0(\omega, \rho) \equiv 1.$$

$n = 1$

$$\begin{aligned} k_1(\theta, \omega, \rho) = & \sin \theta + D_{1,3}^{(1)} \sin 3\theta + \omega^2 [D_{2,3}^{(1)} \sin 3\theta + D_{2,5}^{(1)} \sin 5\theta] + \\ & + \omega^3 [D_{3,3}^{(1)} \sin 3\theta + D_{3,5}^{(1)} \sin 5\theta + D_{3,7}^{(1)} \sin 7\theta] + \\ & + \omega^4 [D_{4,3}^{(1)} \sin 3\theta + D_{4,5}^{(1)} \sin 5\theta + D_{4,7}^{(1)} \sin 7\theta + D_{4,9}^{(1)} \sin 9\theta] + O(\omega^5). \end{aligned}$$

For $l_1(\theta, \omega, \rho)$ replace sines by cosines and $D_{m,r}^{(1)}$ by $E_{m,r}^{(1)}$ throughout. Then

$$\begin{aligned}
D_{1,3}^{(1)} = E_{1,3}^{(1)} &= \frac{\rho}{2^4}, & D_{2,3}^{(1)} = -E_{2,3}^{(1)} &= \frac{5\rho^2 + 2}{2^3}, \\
D_{2,5}^{(1)} = E_{2,5}^{(1)} &= -\frac{\rho^2 - 2}{3 \cdot 2^8}, & D_{3,3}^{(1)} = E_{3,3}^{(1)} &= -\frac{\rho(35\rho^2 + 44)}{3 \cdot 2^{12}}, \\
D_{3,5}^{(1)} = -E_{3,5}^{(1)} &= \frac{\rho(2\rho^2 - 1)}{3 \cdot 2^{10}}, & D_{3,7}^{(1)} = E_{3,7}^{(1)} &= \frac{\rho(\rho^2 - 8)}{3^2 \cdot 2^{13}}, \\
D_{4,3}^{(1)} = -E_{4,3}^{(1)} &= -\frac{\rho^2(12\rho^2 + 2)}{3 \cdot 2^{16}}, & D_{4,5}^{(1)} = E_{4,5}^{(1)} &= -\frac{\rho^2(343\rho^2 + 316)}{3^3 \cdot 2^{14}}, \\
D_{4,7}^{(1)} = -E_{4,7}^{(1)} &= \frac{61\rho^4 - 308\rho^2 - 42}{3^3 \cdot 2^{18}}, & D_{4,9}^{(1)} = E_{4,9}^{(1)} &= \frac{\rho^2 - 20\rho^2 + 24}{5 \cdot 3^2 \cdot 2^{18}}.
\end{aligned}$$

Also

$$\begin{aligned}
C_1(\omega, \rho) &= -\frac{\omega\rho}{2} + \frac{\rho(3\rho^2 + 4)}{29} \omega^3 - \frac{\rho^2(3\rho^2 + 2)}{2^{12}} \omega^4 + O(\omega^5), \\
S_1(\omega, \rho) &= -\frac{\omega\rho}{2} + \frac{\rho(3\rho^2 + 4)}{29} \omega^3 + \frac{\rho^2(3\rho^2 + 2)}{2^{12}} \omega^4 + O(\omega^5).
\end{aligned}$$

$n = 2$

$$\begin{aligned}
k_2(\theta, \omega, \rho) &= \sin 2\theta + \omega D_{1,4}^{(2)} \sin 4\theta + \omega^2 D_{2,6}^{(2)} \sin 6\theta + \\
&\quad + \omega^3 [D_{3,4}^{(2)} \sin 4\theta + D_{3,8}^{(2)} \sin 8\theta] + \\
&\quad + \omega^4 [D_{4,6}^{(2)} \sin 6\theta + D_{4,10}^{(2)} \sin 10\theta] + O(\omega^5), \\
l_2(\theta, \omega, \rho) &= \cos 2\theta + \omega [E_{1,0}^{(2)} + E_{1,4}^{(2)} \cos 4\theta] + \omega^2 E_{2,6}^{(2)} \cos 6\theta + \\
&\quad + \omega^3 [E_{3,0}^{(2)} + E_{3,4}^{(2)} \cos 4\theta + E_{3,8}^{(2)} \cos 8\theta] + \\
&\quad + \omega^4 [E_{4,6}^{(2)} \cos 6\theta + E_{4,10}^{(2)} \cos 10\theta] + O(\omega^5),
\end{aligned}$$

where

$$\begin{aligned}
E_{1,0}^{(2)} &= -\frac{1}{2^2}, & D_{1,4}^{(2)} = E_{1,4}^{(2)} &= \frac{\rho}{3 \cdot 2^3}, & D_{2,6}^{(2)} = E_{2,6}^{(2)} &= \frac{\rho^2 - 3}{3 \cdot 2^9}, \\
E_{3,0}^{(2)} &= -\frac{\rho(\rho^2 + 2)}{3 \cdot 2^9}, & D_{3,4}^{(2)} &= -\frac{\rho(53\rho^2 + 81)}{3^3 \cdot 2^{12}}, & E_{3,4}^{(2)} &= \frac{\rho(91\rho^2 + 207)}{3^3 \cdot 2^{12}}, \\
D_{3,8}^{(2)} = E_{3,8}^{(2)} &= \frac{\rho(\rho^2 - 11)}{5 \cdot 3^2 \cdot 2^{12}}, & D_{4,6}^{(2)} &= -\frac{454\rho^4 + 48\rho^2 - 675}{5 \cdot 3^2 \cdot 2^{18}}, \\
E_{4,6}^{(2)} &= \frac{413\rho^4 + 462\rho^2 - 675}{5 \cdot 3^3 \cdot 2^{18}}, & D_{4,10}^{(2)} = E_{4,10}^{(2)} &= \frac{\rho^4 - 26\rho^2 + 45}{5 \cdot 3^3 \cdot 2^{18}}.
\end{aligned}$$

Also

$$C_2(\omega, \rho) = \frac{\rho^2 + 1}{2^5} \omega^2 - \frac{17\rho^2(\rho^2 + 1)}{3^2 \cdot 2^{11}} + O(\omega^5),$$

$$S_2(\omega, \rho) = \frac{\rho^2 + 1}{2^5} \omega^2 + \frac{\rho^2(\rho^2 + 1)}{3^2 \cdot 2^{11}} \omega^4 + O(\omega^5).$$

4. Limiting Cases of the Whittaker-Hill Equation

The Whittaker-Hill equation reduces to Mathieu's equation in two ways.

(i) Let $\omega \rightarrow 0$, $\rho \rightarrow \infty$ in such a way that $\omega\rho \rightarrow -2q$, q finite. Then equation (1.1) becomes

$$(4.1) \quad \frac{d^2\Phi}{d\theta^2} + (\eta - 2q \cos 2\theta)\Phi = 0,$$

which is Mathieu's equation.

The perturbation forms of $gc_n(\theta, \omega, \rho)$ and $gs_n(\theta, \omega, \rho)$ reduce directly to the perturbation forms of the Mathieu functions $ce_n(\theta, q)$, $se_n(\theta, q)$ respectively. Similarly the perturbation forms of the second solutions $kc_n(\theta, \omega, \rho)$, $ls_n(\theta, \omega, \rho)$ reduce to the second solutions $fe_n(\theta, q)$, $ge_n(\theta, q)$ of Mathieu's equation. In McLachlan's notation [3] these solutions are

$$fe_n(\theta, q) = C_n(q) \theta ce_n(\theta, q) + f_n(\theta, q),$$

$$ge_n(\theta, q) = S_n(q) \theta se_n(\theta, q) + g_n(\theta, q),$$

where $f_n(\theta, q)$ is odd in θ with the same period as $ce_n(\theta, q)$, and $g_n(\theta, q)$ is even in θ with the same period as $se_n(\theta, q)$. Thus

$$(4.3) \quad \lim_{\omega\rho \rightarrow -2q} C_n(\omega, \rho) = C_n(q), \quad \lim_{\omega\rho \rightarrow -2q} k_n(\theta, \omega, \rho) = f_n(\theta, q),$$

$$(4.4) \quad \lim_{\omega\rho \rightarrow -2q} S_n(\omega, \rho) = S_n(q), \quad \lim_{\omega\rho \rightarrow -2q} l_n(\theta, \omega, \rho) = g_n(\theta, q).$$

These reductions provide a useful check on the highest power of ρ occurring in any coefficient of our perturbation forms, since we can compare them with the corresponding results for Mathieu functions given by McLachlan [3] and Ince [1]. This check is especially helpful as n increases, since the coefficients rapidly become complicated. It should be noted that the coefficient of q^4 in $S_2(q)$ as given by McLachlan and Ince is incorrect: it should be, using McLachlan's notation, 1/1152, not 53/6912. This has been checked independently of our reduction.

(ii) Let $\rho \rightarrow 0$, and put $2\theta = s$. Then equation (1.1) becomes

$$(4.5) \quad \frac{d^2\Phi}{ds^2} + \left(\frac{8\eta + \omega^2}{32} - \frac{\omega^2}{32} \cos 2s\theta \right) \Phi = 0.$$

Comparing (4.5) with Mathieu's equation

$$(4.6) \quad \frac{d^2\Phi}{ds^2} + (a - 2q \cos 2\theta) \Phi = 0$$

we have

$$\omega^2 = 64q, \quad a = \frac{1}{4} \eta + 2q.$$

This reduction provides a check on the terms independent of ρ in our forms. It gives rise to Mathieu functions of both integral and fractional orders.

If $n = 2m$, $m = 0, 1, 2, \dots$, $gc_{2m}(\theta, \omega, \rho)$, $gs_{2m}(\theta, \omega, \rho)$ reduce to $ce_m(s, q)$, $se_m(s, q)$ respectively. From equations (2.1), (2.2) we then have

$$(4.7) \quad \lim_{\rho \rightarrow 0} kc_{2m}(\theta, \omega, \rho) = \frac{1}{2} sce_m(s, q) \lim_{\rho \rightarrow 0} C_{2m}(\omega, \rho) + \lim_{\rho \rightarrow 0} k_{2m}(\theta, \omega, \rho),$$

$$(4.8) \quad \lim_{\rho \rightarrow 0} ls_{2m}(\theta, \omega, \rho) = \frac{1}{2} s se_{2m}(s, q) \lim_{\rho \rightarrow 0} S_{2m}(\omega, \rho) + \lim_{\rho \rightarrow 0} l_{2m}(\theta, \omega, \rho).$$

Comparing our forms for (4.7), (4.8) with those for the Mathieu functions, we find that

$$\lim_{\rho \rightarrow 0} k_{2m}(\theta, \omega, \rho) = f_m(s, q), \quad \lim_{\rho \rightarrow 0} l_{2m}(\theta, \omega, \rho) = g_m(s, q),$$

so that

$$\lim_{\rho \rightarrow 0} C_{2m}(\omega, \rho) = C_m(q), \quad \lim_{\rho \rightarrow 0} S_{2m}(\omega, \rho) = S_m(q),$$

and $kc_{2m}(\theta, \omega, \rho) \rightarrow fe_m(s, q)$, $ls_{2m}(\theta, \omega, \rho) \rightarrow ge_m(s, q)$ as $\rho \rightarrow 0$. The only exception to this is when $m = 0$, where $k_0(\theta, \omega, \rho) \rightarrow \frac{1}{2} f_0(s, q)$, and

$$kc_0(\theta, \omega, \rho) \rightarrow \frac{1}{2} fe_0(s, q).$$

When $n = 2m + 1$, $m = 0, 1, 2, \dots$, $gc_{2m+1}(\theta, \omega, \rho)$, $gs_{2m+1}(\theta, \omega, \rho)$ reduce to Mathieu functions of period 4π , namely $ce_{m+\frac{1}{2}}(s, q)$, $se_{m+\frac{1}{2}}(s, q)$ respectively. Now the second solution corresponding to $ce_{m+\frac{1}{2}}(s, q)$ is $se_{m+\frac{1}{2}}(s, q)$, so we expect $C_{2m+1}(\omega, \rho)$ to reduce to zero as $\rho \rightarrow 0$, and $kc_{2m+1}(\theta, \omega, \rho)$ to reduce to $se_{m+\frac{1}{2}}(s, q)$. Similarly, as $\rho \rightarrow 0$, $S_{2m+1}(\omega, \rho) \rightarrow 0$ and $ls_{2m+1}(\theta, \omega, \rho) \rightarrow ce_{m+\frac{1}{2}}(s, q)$. This reduction is clearly illustrated

by the forms quoted for $kc_1(\theta, \omega, \rho)$ and $ls_1(\theta, \omega, \rho)$ in § 3. The constants C_{2m+1} , S_{2m+1} , all have a factor ρ which ensures their reduction to zero as $\rho \rightarrow 0$. The constants C_{2m} , S_{2m} do not, of course, contain such a factor.

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A DUOUA SOLUȚIE (NEPERIODICĂ) A ECUAȚIEI LUI WHITTAKER-HILL PENTRU PARAMETRI MICI

(Rezumat)

Se determină a doua soluție a ecuației lui Whittaker-Hill (1.1), cu ajutorul lui (2.1), (2.2), în care funcția $k_n(\theta, \omega, \rho)$ satisface ecuația (2.3). Se determină soluțiile perturbate pînă la $O(\omega^4)$, pentru $n = 0, 1, \dots, 6$ (se dau în detaliu cele corespunzînd lui $n = 0, 1$ și 2). De asemenea se studiază unele cazuri limită ale ecuației (1.1).

BOUNDARY VALUE PROBLEMS FOR MANGERON'S EQUATIONS (1) ¹ ²)

BY

NEHMET NAMIK OĞUZTÖRELİ

1. Introduction

In a previous paper [1], we considered the Mangeron equation

$$(1.1) \quad \partial[p(x)q(y) \partial u] - \lambda r(x) s(y)u = 0$$

for $a < x < b$, $c < y < d$, subject to the boundary conditions

$$(1.2) \quad u(a, y) = u(b, y) = u(x, c) = u(x, d) = 0,$$

where a, b, c and d are certain finite numbers,

$$(1.3) \quad \partial = \frac{\partial^2}{\partial x \partial y},$$

the „total derivative operator“ of M. Picone [2], λ is a parameter, $p(x)$ and $r(x)$, $q(y)$ and $s(y)$ are certain given functions continuous and positive in $a \leq x \leq b$ and $c \leq y \leq d$, respectively, $p(x)$ and $q(y)$ being continuously differentiable in their domains of definition. It was shown in [1] that the functions

$$(1.4) \quad u_{m,n}(x, y) = U_m(x) V_n(y), \quad (m, n = 1, 2, 3, \dots)$$

are eigenfunctions of the boundary value problem (1.1) — (1.2) with the corresponding eigenvalues

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²) The reader is invited to consult both our joint with prof. D. Mangeron papers as well as the author's very recent paper published in the Rendiconti dell'Accademia Nazionale dei Lincei.

$$(1.5) \quad \lambda = \lambda_{m,n} = \mu_m \nu_n,$$

where $U_m(x)$ and $V_n(y)$ are the normalized solutions (with weights $r(a)$ and $s(y)$, respectively) of the following Sturm-Liouville problem (with eigenvalues $\mu = \mu_m$ and $\nu = \nu_n$) respectively:

$$(1.6) \quad \frac{d}{dx} \left[p(x) \frac{dU}{dx} \right] + \mu r(x) U = 0, \quad U(a) = U(b) = 0,$$

and

$$(1.7) \quad \frac{d}{dy} \left[q(y) \frac{dV}{dy} \right] + \nu s(y) V = 0, \quad V(c) = V(d) = 0.$$

The functions $\{u_{m,n}(x, y)\}$ constitute a complete set of functions orthonormal with weight $r(x)s(y)$ over the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$.

In this paper we consider the Mangeron equation

$$(1.8) \quad \frac{1}{r(x)s(y)} \partial[p(x)q(y)\partial u] - \lambda u = f(x, y), \quad (a < x < b, c < y < d),$$

subject to the boundary conditions (1.2), where p, q, r and s are as above, and $f(x, y)$ is a given function sufficiently smooth on R . For the existence of a solution of the above problem we refer to [4] and [5].

2. Formal Solution

Since the functions $\{u_{m,n}(x, y)\}$ constitute a complete set of functions orthonormal with weight $r(x)s(y)$ over the rectangle R , the function $f(x, y)$ admits the eigenfunction expansion

$$(2.1) \quad f(x, y) = \sum_{m,n=1}^{\infty} a_{m,n} U_m(x) V_n(y)$$

with

$$(2.2) \quad a_{m,n} = \iint_R f(\xi, \eta) r(\xi) s(\eta) U_m(\xi) V_n(\eta) d\xi d\eta.$$

We now assume that Eq. (1.8) admits a solution $u = u(x, y)$ satisfying the boundary conditions (1.2). Clearly we may write

$$(2.3) \quad u(x, y) = \sum_{m,n=1}^{\infty} c_{m,n} U_m(x) V_n(y)$$

where

$$(2.4) \quad c_{m,n} = \iint_R u(\xi, \eta) r(\xi) s(\eta) U_m(\xi) V_n(\eta) d\xi d\eta.$$

We can easily verify that

$$(2.5) \quad \frac{1}{r(x)s(y)} \partial[p(x)q(y)\partial u] = \sum_{m,n=1}^{\infty} \lambda_{m,n} c_{m,n} U_m(x) V_n(y),$$

if term by term differentiations are allowed in (2.3). Under this condition we find the equalities

$$(2.6) \quad (\lambda_{m,n} - \lambda) c_{m,n} = a_{m,n}, \quad (m, n = 1, 2, 3, \dots).$$

Thus, if $\lambda \neq \lambda_{m,n}$,

$$(2.7) \quad c_{m,n} = \frac{a_{m,n}}{\lambda_{m,n} - \lambda}, \quad (m, n = 1, 2, 3, \dots).$$

Hence

$$(2.8) \quad u(x, y) = \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{\lambda_{m,n} - \lambda} U_m(x) V_n(y).$$

If $\lambda = \lambda_{m_1, n_1} = \lambda_{m_2, n_2} = \dots = \lambda_{m_k, n_k}$ and $a_{m_1, n_1} = a_{m_2, n_2} = \dots = a_{m_k, n_k} = 0$,

then we have

$$(2.9) \quad u(x, y) = \sum_{i=1}^k C_i U_{m_i}(x) V_{n_i}(y) + \sum_{m,n=1}^{\infty} \frac{a_{m,n}}{\lambda_{m,n} - \lambda} U_m(x) V_n(y),$$

where $C_1, \dots, C_k$ are arbitrary constants and $\sum'$ denotes the summation over the indices m and n , the pairs $(m, n) = (m_i, n_i)$ ($i = 1, \dots, k$) being excluded. Clearly, if $\lambda = \lambda_{m_0, n_0}$ and $a_{m_0, n_0} \neq 0$ there is no solution for our problem.

Thus, the boundary value problem (1.8), (1.2) admits a unique solution of the form (2.8) if $\lambda \neq \lambda_{m,n}$ and if term by term differentiation are allowed in (2.8), and infinitely many solutions of the form (2.9) with arbitrary C_i ($i = 1, \dots, k$) if $\lambda = \lambda_{m_1, n_1} = \lambda_{m_2, n_2} = \dots = \lambda_{m_k, n_k}$ and $a_{m_1, n_1} = a_{m_2, n_2} = \dots = a_{m_k, n_k} = 0$, no solution if $\lambda = \lambda_{m_0, n_0}$ and $a_{m_0, n_0} \neq 0$.

3. Integral Representation of the Solution

Assume that $\lambda \neq \lambda_{m,n}$ and consider the solution $u(x, y)$ given by the formula (2.8). Substituting the expressions (2.2) into (2.8), and assuming the uniform convergence of the series under consideration, we find

$$\begin{aligned}
 u(x, y) &= \sum_{m,n=1}^{\infty} \frac{U_m(x)V_n(y)}{\lambda_{m,n} - \lambda} \left\{ \iint_R f(\xi, \eta) r(\xi) s(\eta) U_m(\xi) V_n(\eta) d\xi d\eta \right\} = \\
 &= \iint_R \left\{ \sum_{m,n=1}^{\infty} \frac{U_m(x)V_n(y)U_m(\xi)V_n(\eta)}{\lambda_{m,n} - \lambda} \right\} r(\xi)s(\eta)f(\xi, \eta) d\xi d\eta.
 \end{aligned}$$

Thus, putting

$$(3.1) \quad G(x, y; \xi, \eta) = \sum_{m,n=1}^{\infty} \frac{U_m(x)V_n(y)U_m(\xi)V_n(\eta)}{\lambda_{m,n} - \lambda},$$

we obtain the following integral representation of the solution :

$$(3.2) \quad u(x, y) = \iint_R G(x, y; \xi, \eta) r(\xi) s(\eta) f(\xi, \eta) d\xi d\eta.$$

Remark. In a number of problems, the conditions $p > 0$, $q > 0$, $r > 0$ and $s > 0$ on R can be relaxed by the inequalities $p \geq 0$, $q \geq 0$, $r \geq 0$ and $s \geq 0$ on R , respectively.

4. Example

We now illustrate the above results by the following simple example. Consider the polyvibrating equation (cf. [3] — [6]) :

$$(4.1) \quad \partial^2 u - \lambda u = f(x, y)$$

for $0 < x, y < \pi$, subject to the boundary conditions

$$(4.2) \quad u|_{x=0} = u|_{x=\pi} = u|_{y=0} = u|_{y=\pi} = 0.$$

In this case we have

$$(4.3) \quad u_{m,n}(x, y) = \frac{2}{\pi} \sin mx \sin ny, \quad \lambda_{m,n} = m^2 n^2, \quad (m, n = 1, 2, 3, \dots),$$

and therefore, the series (2.1) and (2.3) are the double sine Fourier series of the functions $f(x, y)$ and $u(x, y)$, respectively.

If the function $f(x, y)$ is continuously twice differentiable on $R = \{(x, y) | 0 \leq x, y \leq \pi\}$, the series (2.1) converges absolutely and uniformly in R , and represents $f(x, y)$ in R . Further, if $\lambda \neq m^2 n^2$, ($m, n = 1, 2, 3, \dots$) the series (2.8) converges absolutely and uniformly on R and we can differentiate twice term by term with respect to x and y . It is the actual solution of the boundary value problem (4.1) — (4.2).

Clearly the series

$$(4.4) \quad G \equiv G(x, y; \xi, \eta) = \sum_{m,n=1}^{\infty} \frac{\sin mx \sin ny \sin m\xi \sin n\eta}{m^2 n^2 - \lambda}$$

is absolutely and uniformly convergent on R , and we have

$$(4.5) \quad \partial^2 G - \lambda G = \delta(x - \xi) \delta(y - \eta),$$

where $\delta(t)$ is the Dirac's delta. Accordingly we have the representation formula :

$$(4.6) \quad u(x, y) = \iint_R G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

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PROBLEME PE CONTUR PENTRU ECUAȚII MANGERON (I)

(Rezumat)

Se studiază ecuația (1.8) cu condițiile la limită (1.2), obținind expresia formală a soluției precum și o reprezentare integrală a acesteia (3.2).

in this respect, the American Medical Association is in a position to do much to help the public. It is the duty of the American Medical Association to do this, and it is the duty of the American Medical Association to do this in a way that will be to the benefit of the public.

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THE DARBOUX PROBLEM FOR A HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER

BY

GEORGETA TEODORU

The aim of this note is to consider the Darboux problem :

$$(1) \quad \begin{aligned} z_{xy} &= f(x, y, z), & z(x, 0) &= \sigma(x), \\ z(0, y) &= \tau(y), & \sigma(0) &= \tau(0) = z_0, \end{aligned}$$

where $(x, y) \in R^2$ and to establish the existence of a solution of (1) using the Tychonoff's fixed point theorem [1], [2], that is :

If E is a complete locally convex linear space, $T: E \rightarrow E$ a continuous operator, A a convex closed subset of E , $T(A) \subset A$ and $\overline{T(A)}$ compact, then T has a fixed point belonging to A .

It is well known that if T is a compact operator and A is a bounded set, then $\overline{T(A)}$ is compact ; thus we obtain the following corollary :

Under the hypotheses of Tychonoff's fixed point theorem, A being bounded and T compact, it results that T has a fixed point belonging to A .

There exists [1] a bijection of the set of solutions of problem (1) and the set of fixed points of the operator T defined by :

$$(2) \quad (Tz)(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(\xi, \eta, z(\xi, \eta)) d\eta d\xi,$$

$$(x, y) \in R^2,$$

where $z \in C(R^2, R)$, the space $C(R^2, R)$ being a complete locally convex space [1].

The operator T given by (2) is continuous and compact [1] in topology of $C(R^2, R)$.

which is uniformly convergent to a bounded function $g \in C(R^2, R^+)$, because we can write :

$$\begin{aligned} |g_1(x, y) - g_0(x, y)| &= |\alpha(x, y)| = \alpha(x, y) \leq M, \\ |g_2(x, y) - g_1(x, y)| &= \left| \iint_{00}^{xy} \varphi(\xi, \eta, \alpha(\xi, \eta)) \, d\eta \, d\xi \right| = \\ &= \left| \iint_{00}^{xy} [\varphi(\xi, \eta, \alpha(\xi, \eta)) - \varphi(\xi, \eta, 0)] \, d\eta \, d\xi \right| \leq \\ &\leq \left| \iint_{00}^{xy} \psi(\xi, \eta) |\alpha(\xi, \eta)| \, d\eta \, d\xi \right| \leq M \left| \iint_{00}^{xy} \psi(\xi, \eta) \, d\eta \, d\xi \right| \leq MI, \\ |g_3(x, y) - g_2(x, y)| &\leq \left| \iint_{00}^{xy} \psi(\xi, \eta) |g_2(\xi, \eta) - g_1(\xi, \eta)| \, d\eta \, d\xi \right| \leq \\ &\leq MI \left| \iint_{00}^{xy} \psi(\xi, \eta) \, d\eta \, d\xi \right| \leq MI^2, \\ &\dots \dots \dots \\ |g_n(x, y) - g_{n-1}(x, y)| &\leq \left| \iint_{00}^{xy} \psi(\xi, \eta) |g_{n-1}(\xi, \eta) - g_{n-2}(\xi, \eta)| \, d\eta \, d\xi \right| \leq \\ &\leq MI^{n-2} \left| \iint_{00}^{xy} \psi(\xi, \eta) \, d\eta \, d\xi \right| \leq MI^{n-1}, \\ &\dots \dots \dots \end{aligned}$$

From $I < 1$ it results that the series

$$\sum_{n=1}^{\infty} (g_n - g_{n-1})$$

is uniformly convergent for $(x, y) \in R^2$, or equivalently, the sequence of successive approximations is uniformly convergent for $(x, y) \in R^2$.

Thus, there exists $g \in C(R^2, R^+)$ such that

$$g(x, y) = \lim_{n \rightarrow \infty} g_n(x, y)$$

uniformly for $(x, y) \in R^2$.

From

$$g_n(x, y) = \alpha(x, y) + \left| \iint_{00}^{xy} \varphi(\xi, \eta, g_{n-1}(\xi, \eta)) d\eta d\xi \right|$$

it results the function g is a solution of (3), and it is bounded by the sum of the series $\sum_{n=1}^{\infty} MI^{n-1}$.

The unicity of the solution of (3) is proved by reductio ad absurdum; supposing that (3) has also the solution $h \in C(R^2, R^+)$ such that

$$H = \sup |g(x, y) - h(x, y)| > 0, \quad (x, y) \in R^2$$

it results

$$|g(x, y) - h(x, y)| \leq \left| \iint_{00}^{xy} \psi(\xi, \eta) |g(\xi, \eta) - h(\xi, \eta)| d\eta d\xi \right| \leq HI$$

and also the contradiction $H \leq IH$.

Let $A \subset C(R^2, R)$, be the subset defined by

$$A = \{z \in C(R^2, R) \mid |z(x, y)| \leq g(x, y), (x, y) \in R^2\}$$

where g is the solution of (3).

The set A is closed convex and bounded [1] in the topology of $C(R^2, R)$. The operator T defined by (2) applies A into A ; in fact, if $z \in A$, then using $1^\circ, 2^\circ$, it results:

$$\begin{aligned} |(Tz)(x, y)| &\leq |\sigma(x) + \tau(y) - z_0| + \left| \iint_{00}^{xy} f(\xi, \eta, z(\xi, \eta)) d\eta d\xi \right| \leq \\ &\leq \alpha(x, y) + \left| \iint_{00}^{xy} \varphi(\xi, \eta, g(\xi, \eta)) d\eta d\xi \right| \end{aligned}$$

therefore

$$|(Tz)(x, y)| \leq g(x, y), \quad \text{for } (x, y) \in R^2,$$

that is $Tz \in A$.

Using the corollary of Tychonoff's fixed point theorem, it results the operator T has a fixed point belonging to A , that is the problem (1) has a solution into A (Q. E. D.).

Remark. If we replace the condition $\varphi(x, y, 0) = 0$ by

$$\sup_{Dxy} \iint \varphi(\xi, \eta, \alpha(\xi, \eta)) d\eta d\xi = C_0, \quad (x, y) \in R^2,$$

we obtain the same result as above.

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PROBLEMA LUI DARBOUX PENTRU O ECUAȚIE CU DERIVATE PARȚIALE DE ORDINUL DOI, DE TIP HIPERBOLIC

(Rezumat)

Se demonstrează o teoremă de existență a unei soluții pentru problema lui Darboux (1), utilizând în acest scop o teoremă de punct fix a lui Tihonov.

ON THE ZEROS OF SOLUTIONS OF MANGERON'S EQUATIONS ¹⁾

BY

SUNDARAM EASWARAN

1. Introduction

Many comparison theorems for ordinary differential equations have been extended to partial differential equations of elliptic parabolic type by Swanson C. A. [1], Allegretto W. [2], K. Kreith [3]. In this note our aim is to show that some of these results can also be extended to Mangeron's equations. Specifically we will consider the equation

$$(1) \quad \frac{\partial^4 u}{\partial x^2 \partial y^2} = \lambda p(x, y)u$$

subject to the boundary conditions

$$(2) \quad \alpha_i u(x_i, y) + (-1)^i u_x(x_i, y) = 0, \quad \beta_i u(x, y_i) + (-1)^i u_y(x, y_i) = 0 \\ (i = 1, 2)$$

and the compatibility conditions

$$(3) \quad u_x(x_1, y_1) u_x(x_2, y_2) u_y(x_1, y_2) u_y(x_2, y_1) = \\ = u_x(x_1, y_2) u_x(x_2, y_1) u_y(x_1, y_1) u_y(x_2, y_2)$$

where $u(x, y)$ is a continuous function defined on $R: \{x_1 \leq x \leq x_2; y_1 \leq y \leq y_2\}$ such that $u_x(x, y)$, $u_y(x, y)$, $u_{xy}(x, y)$, $u_{x^2y}(x, y)$, $u_{xy^2}(x, y)$, $u_{x^2y^2}(x, y)$ are all continuous in R and α_i, β_i are non negative constants such that at least one of the products $\{\alpha_i \beta_j\}_{i,j=1,2}$ is not zero. Consider the relationship

¹⁾ The preparation of this paper was assisted by the Grant NRC-4345 from the National Research Council of Canada.

$$\begin{aligned}
 (4) \quad \int_{x_1}^{x_2} \int_{y_1}^{y_2} u \frac{\partial^4 u}{\partial x^2 \partial y^2} dx dy &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} u_{xy}^2 dx dy + \int_{x_1}^{x_2} \sum_{i=1}^2 \beta_i u_x^2(x, y_i) dx + \\
 &+ \int_{y_1}^{y_2} \sum_{i=1}^2 \alpha_i u_y^2(x_i, y) dy + \sum_{i,j=1}^2 \alpha_i \beta_j u^2(x_i, y_j),
 \end{aligned}$$

which can easily be verified [4]. Since (4) gives a positive contribution in the calculation of the smallest eigenvalue, we get the positivity of the smallest eigenvalue of the boundary value problem (1) – (3).

The Green's function for the operator

$$Lu = \frac{\partial^4 u}{\partial x^2 \partial y^2}$$

subject to the boundary conditions (2), (3) is given by

$$(5) \quad G(x, y; \xi, \eta) = G_1(x; \xi) G_2(y; \eta),$$

where $G_1(x, \xi)$, $G_2(y; \eta)$ are respectively the Green's functions of the following problems:

$$(6) \quad \begin{cases} \Phi''(x) = f(x), & \alpha_i \Phi(x_i) + (-1)^i \Phi'(x_i) = 0, & (i = 1, 2), \\ \Psi''(y) = g(y), & \beta_i \Psi(y_i) + (-1)^i \Psi'(y_i) = 0, & (i = 1, 2). \end{cases}$$

The sign of $G_1(x; \xi)$ and $G_2(y; \eta)$ depend only on the signs of the constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ and hence $G_1(x; \xi)$ and $G_2(y; \eta)$ have the same sign and therefore $G(x, y; \xi, \eta)$ is always positive. Accordingly by Jentzsch Theorem [6] the eigenfunction corresponding to the smallest eigenvalue is positive in R . From the orthogonality of eigenfunctions we conclude that the other eigenfunctions must have a zero in R . We now prove the following

Theorem. Let $u(x, y)$ be a solution of the problem

$$(7) \quad \frac{\partial^4 u}{\partial x^2 \partial y^2} = p(x, y)u,$$

$$(8) \quad \alpha_i u(x_i, y) + (-1)^i u_x(x_i, y) = 0, \quad \beta_i u(x, y_i) + (-1)^i u_y(x, y_i) = 0, \\
 (i = 1, 2)$$

subject to the compatibility conditions (3) and let $v(x, y)$ be a solution of the problem

$$(9) \quad \frac{\partial^4 v}{\partial x^2 \partial y^2} = q(x, y)v,$$

$$(10) \quad \gamma_i v(x_i, y) + (-1)^i v_x(x_i, y) = 0, \quad \delta_i v(x, y_i) + (-1)^i v_y(x, y_i) = 0, \\ (i = 1, 2)$$

subject to the compatibility conditions (3). If $p(x, y)$ and $q(x, y)$ are continuous functions such that $q(x, y) \geq p(x, y) > 0$ and $0 < \gamma_i < \alpha_i$, $0 < \delta_j < \beta_j$ ($i, j = 1, 2$) then $v(x, y)$ has a zero in $(x_1 < x < x_2; y_1 < y < y_2)$.

P r o o f. It can easily be seen from (7), (8) that $u(x, y)$ is an eigenfunction corresponding to $\lambda = 1$ of the problem

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} + (q - p)u = \lambda qu$$

and similarly $v(x, y)$ is an eigenfunction corresponding to $\mu = 1$ of the problem

$$\frac{\partial^4 v}{\partial x^2 \partial y^2} = \mu qv,$$

subject to the respective boundary conditions. If $v(x, y) \neq 0$ in R , then the smallest eigenvalue of (7) and (8) is equal to $\mu_1 = 1$. If we let

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} p u^2(x, y) dx dy = 1$$

and since $u(x, y)$ does not satisfy the boundary conditions (10), the minimal property of the first eigenfunction yields

$$\begin{aligned} \mu_1 &< \int_{x_1}^{x_2} \int_{y_1}^{y_2} u_{xy}^2 dx dy + \sum_{i,j=1}^2 \gamma_i \delta_j u^2(x_i, y_j) + \\ &+ \int_{x_1}^{x_2} \sum_{i=1}^2 \delta_i u_x^2(x, y_i) dx + \int_{y_1}^{y_2} \sum_{i=1}^2 \gamma_i u_y^2(x_i, y) dy \leq \\ &\leq \int_{x_1}^{x_2} \int_{y_1}^{y_2} \{u_{xy}^2(x, y) + (q-p)u^2\} dx dy + \sum_{i,j=1}^2 \alpha_i \beta_j u^2(x_i, y_j) + \\ &+ \int_{x_1}^{x_2} \sum_{i=1}^2 \beta_i u_x^2(x, y_i) dx + \int_{y_1}^{y_2} \sum_{i=1}^2 \alpha_i u_y^2(x_i, y) dy = 1, \end{aligned}$$

which leads to a contradiction since $\mu_1 = 1$.

Corollary. If $\alpha_i = \infty$, $\beta_i = \infty$ and $u(x, y)$ and $v(x, y)$ are respectively solutions of (7) and (10) and $u(x, y)$ satisfies the boundary conditions $u(x_i, y) = 0$, $u(x, y_i) = 0$ ($i = 1, 2$) and $v(x, y)$ satisfies (11) and (3), then $v(x, y)$ has a zero in R .

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ASUPRA ZEROURILOR SOLUȚIILOR ECUAȚIILOR LUI MANGERON

(Rezumat)

Se consideră problema la limită (2), (3) pentru ecuația (1) și se discută zerourile soluției.

OPTIMIZATION OF CONTROL SYSTEMS DESCRIBED
BY VOLTERRA INTEGRAL EQUATIONS

BY

C. F. LEE

1. Consider a control system whose state at time t is described by an n -dimensional real-valued vector function $x(y) = (x_1(t), x_2(t), \dots, x_n(t))$. Let R be a compact and convex set in the r -dimensional euclidean space E^r . We assume that the control at time t is represented by an r -dimensional real-valued vector function $u(t) = (u_1(t), u_2(t), \dots, u_r(t))$ such that $u(t) \in R$ for any t in a specified time interval.

The state of the system at the initial time $t = t_0$ is denoted by x_0 . It is assumed that $x_0 \in G \subset E^n$, when G is a compact set containing the origin of the space E^n .

Let $[t_1, t_2]$, where $t_0 \leq t_1 \leq t_2$, be a compact time interval. We assume that the control system is governed by a linear Volterra integral equation of the second kind:

$$(1) \quad x(t) = A(t) u(t) + \int_{t_0}^t K(t, \tau) x(\tau) d\tau, \quad t \in [t_0, t_2],$$

where $A(t) = \{A_{ij}(t)\}$, ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, r$) is a continuous $n \times r$ matrix, $K(t, \tau) = \{K_{pq}(t, \tau)\}$, ($p = 1, 2, \dots, n$; $q = 1, 2, \dots, n$) is a continuous $n \times n$ matrix.

A continuous vector function $u(t)$ defined for $t \in [t_0, \bar{t}]$, $\bar{t} \in [t_1, t_2]$, with its range in the region R , will be called an admissible control if it satisfies the condition

$$(2) \quad A(t_0) u(t_0) = x_0.$$

The set of all admissible control will be denoted by U . The set U is assumed to be compact, containing $u(t) = 0$ as an interior point.

The Volterra integral equation (1) for an [1], [2], [3] admissible control $u(t)$ has a unique solution α which will be denoted by $x(t, t_0, x_0; u)$, or $x(t)$. Note that

$$(3) \quad x(t) = A(t)u(t) + \int_{t_0}^t H(t, \tau)A(\tau)d\tau, \quad t \in [t_0, t_2],$$

where $H(t, \tau) = \{H_{kl}(t, \tau)\}$, ($k = 1, 2, \dots, n$; $l = 1, 2, \dots, n$) is a continuous $n \times n$ matrix, which represents the resolvent of the integral equation (1).

It is assumed that the cost functional $J(x_0; u)$ measuring the performance of the control system is of the form

$$(4) \quad J(x_0; u) = \int_{t_0}^{\bar{t}} g(x(t), u(t); t) dt,$$

where g is a given function which is sufficiently smooth in all of its arguments, $x(t)$ is the solution of equation (1) under the admissible control $u(t)$ defined for $t \in [t_0, \bar{t}]$, which transfers the control system from the initial state $x_0 = x(t_0)$ to the terminal state $\bar{x} = x(\bar{t})$, where t_0 and $\bar{t}$ are the initial and the terminal time respectively.

Let T be a family of closed sets T_t in E^n , defined for $t \in [t_1, t_2]$.

We say that control $u(t)$ transfers the initial state x_0 onto the set T if the solution $x(t, t_0, x_0; u)$ corresponding to the policy $\{x_0, u\}$ satisfies the relations

$$(5) \quad x(\bar{t}, t_0, x_0; u) \in T_{\bar{t}}.$$

In this article we deal with the following optimization problem:

Find an admissible control $\bar{u}(t)$ which transfers the control system from the initial state x_0 onto the set T with a minimal cost, i. e.,

$$(6) \quad J(x_0; \bar{u}) = \min_{u \in U} \int_{t_0}^{\bar{t}} g(x(t), u(t); t) dt.$$

2. Consider the set of points $(t, x, \varphi) \in E^{n+2}$, $t \in [t_0, t_2]$, $x \in E^n$, and $\varphi \in [0, \infty)$. A point $(\tilde{t}, \tilde{x}, \tilde{\varphi}) \in E^{n+2}$ is said to be attainable if there exists an admissible control $\tilde{u}(t)$ such that equation (1) with the initial condition

$$(7) \quad \tilde{x}(t_0) = x_0 \in G$$

has the unique solution

$$(8) \quad \tilde{x}(t) = A(t)\tilde{u}(t) + \int_{t_0}^t H(t, \tau)A(\tau)\tilde{u}(\tau) d\tau$$

for $t \in [t_0, \tilde{t}]$, $\tilde{t} \in [t_1, t_2]$, satisfying

$$(9) \quad \tilde{x}(\tilde{t}) = \tilde{x}$$

and

$$(10) \quad \tilde{\varphi} = \int_{t_0}^t g(\tilde{x}(t), \tilde{u}(t); t) dt.$$

The set of all attainable points will be called the attainable set for the control system and will be denoted by $\mathcal{A}$.

We have the following theorems which are extensions of the results due to E. Roxin [4] and L. W. Neustadt [5].

Theorem 1. *Under the conditions stated in the last section, the attainable set $\mathcal{A}$ is compact.*

Theorem 2. *In addition to the conditions stated in Theorem 1, if*

(a) *the closed sets T_t are upper semi-continuous with respect to inclusion, and*

(b) *there is at least one admissible pair $\{x_0, u(t)\}$ transferring x_0 onto the set T ,*

then there is an optimal pair $\{x_0, \bar{u}(t)\}$ such that $\bar{u}(t)$ transfers x_0 onto the set T with minimal cost.

3. By using the first variations of the cost functional (4), we can easily derive the following necessary condition for the optimal control [5]:

$$\frac{\partial g(x, u; t)}{\partial x} A(t) + \frac{\partial g(x, u; t)}{\partial u} + \int_t^{\tilde{t}} \frac{\partial g(x, u; t)}{\partial x} H(\sigma, t) A(t) d\sigma = 0,$$

an integral equation with respect to $u = u(t)$.

4. The above results can be extended to control systems described by Volterra integral equations of the form [3]

$$x(t) = A(t)u(t) + f(t) + \int_{t_0}^t K(t, \sigma)x(\sigma) d\sigma, \quad t \in [t_0, t_2],$$

where $f(t)$ is a given n -dimensional continuous vector function.

5. During the past decade a great deal of investigation has been made on control systems described by ordinary differential equations. A detailed study of control systems described by difference-differential equations was presented in [6]. However very few papers [7] were dealing with control systems described by integral equations.

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OPTIMIZAREA SISTEMELOR DE CONTROL DESCRISE DE ECUAȚII INTEGRALE TIP VOLTERRA

(Rezumat)

Se studiază existența soluțiilor problemelor de control optimal descrise de ecuații integrale liniare tip Volterra și se stabilesc condițiile necesare pentru realizarea de controale optimale.

A FUNCTIONAL EQUATION OF VOLTERRA TYPE

BY

ADRIAN CORDUNEANU

1. In this note we shall study the following functional equation

$$(1) \quad u(x, y) = \alpha(x, y) + \int_0^x \int_0^y F\left(x, y, \xi, \eta, u(\xi, \eta), \frac{\partial u}{\partial \xi}(\xi, \eta), \frac{\partial u}{\partial \eta}(\xi, \eta)\right) d\xi d\eta,$$

$$x \geq 0, \quad y \geq 0$$

where α and F are given and u is the unknown function.

Definition. We say that a function $F = F(x, y, \xi, \eta, u, v, w)$ defined on $D = \{(x, y, \xi, \eta, u, v, w); 0 \leq \xi \leq x < \infty, 0 \leq \eta \leq y < \infty, u, v, w \in \mathbb{R}\}$ satisfies the (L) condition if for every $A > 0$ there is a constant $L = L(A)$ such that if $0 \leq \xi \leq x \leq A, 0 \leq \eta \leq y \leq A$ it follows

$$|F(x, y, \xi, \eta, u, v, w) - F(x, y, \xi, \eta, \bar{u}, \bar{v}, \bar{w})| \leq L(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|), \quad \forall u, v, w, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}.$$

We suppose that α belongs to the class C^1 and that F, F_x, F_y are continuous in D and satisfy the (L) condition.

Theorem 1. *If the conditions cited above are satisfied, the equation (1) has a unique solution $u = u(x, y)$ in the class C^1 .*

Proof. We shall make the proof using the method of the successive approximations. We put

After a simple computation, we obtain for $0 \leq x \leq A$, $0 \leq y \leq A$

$$(6) \quad \begin{aligned} |u_2(x, y) - u_1(x, y)| &\leq 3MLK \frac{(x+y)^2}{2!}, \quad \left| \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x} \right| \leq 3MLK (x+y), \\ \left| \frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y} \right| &\leq 3MLK (x+y). \end{aligned}$$

Taking into account the relations (2), (3) and (6) we obtain also for $0 \leq x \leq A$, $0 \leq y \leq A$ that

$$(7) \quad \begin{aligned} |u_3(x, y) - u_2(x, y)| &\leq 3ML^2 K^2 \frac{(x+y)^3}{3!}, \quad \left| \frac{\partial u_3}{\partial x} - \frac{\partial u_2}{\partial x} \right| \leq \\ &\leq 2(3ML^2 K^2) \frac{(x+y)^2}{2!}, \quad \left| \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial y} \right| \leq 2(3ML^2 K^2) \frac{(x+y)^2}{2!}. \end{aligned}$$

We now suppose that for a natural number $n \geq 2$ are true in the domain $0 \leq x \leq A$, $0 \leq y \leq A$ the following relations

$$(8) \quad \left\{ \begin{aligned} |u_{n+1}(x, y) - u_n(x, y)| &\leq 2^{n-2}(3ML^n K^n) \frac{(x+y)^{n+1}}{(n+1)!}, \quad \left| \frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x} \right| \leq \\ &\leq 2^{n-1}(3ML^n K^n) \frac{(x+y)^n}{n!}, \quad \left| \frac{\partial u_{n+1}}{\partial y} - \frac{\partial u_n}{\partial y} \right| \leq 2^{n-1}(3ML^n K^n) \frac{(x+y)^n}{n!}. \end{aligned} \right.$$

If we show that the corresponding relations are true for the natural number $n+1$, it follows that (8) are true for every natural number ≥ 2 . We obtain

$$(9) \quad \begin{aligned} |u_{n+2}(x, y) - u_{n+1}(x, y)| &\leq \iint_{00}^{xy} L |u_{n+1}(\xi, \eta) - u_n(\xi, \eta)| + \\ &+ \left| \frac{\partial u_{n+1}}{\partial \xi} - \frac{\partial u_n}{\partial \xi} \right| + \left| \frac{\partial u_{n+1}}{\partial \eta} - \frac{\partial u_n}{\partial \eta} \right| d\xi d\eta \leq \\ &\leq 2^{n-2}(3ML^{n+1}K^n) \iint_{00}^{xy} \left[\frac{(\xi+\eta)^{n+1}}{(n+1)!} + 4 \frac{(\xi+\eta)^n}{n!} \right] d\xi d\eta \leq \\ &\leq 2^{n-2}(3ML^{n+1}K^n) \frac{(x+y)^{n+2}}{(n+2)!} \left(\frac{x+y}{n+3} + 4 \right) \leq 2^{n-1}(3ML^{n+1}K^{n+1}) \frac{(x+y)^{n+2}}{(n+2)!} \end{aligned}$$

and similarly

$$(10) \quad \left\{ \begin{aligned} & \left| \frac{\partial u_{n+2}}{\partial x}(x, y) - \frac{\partial u_{n+1}}{\partial x}(x, y) \right| \leq \int_0^y L \left(|u_{n+1}(x, \eta) - u_n(x, \eta)| + \right. \\ & \quad \left. + \left| \frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x} \right| + \left| \frac{\partial u_{n+1}}{\partial \eta} - \frac{\partial u_n}{\partial \eta} \right| \right) d\eta + \\ & \quad + \iint_{00}^{xy} L \left(|u_{n+1}(\xi, \eta) - u_n(\xi, \eta)| + \left| \frac{\partial u_{n+1}}{\partial \xi} - \frac{\partial u_n}{\partial \xi} \right| + \right. \\ & \quad \left. + \left| \frac{\partial u_{n+1}}{\partial \eta} - \frac{\partial u_n}{\partial \eta} \right| \right) d\xi d\eta \leq 2^{n-2} (3ML^{n+1}K^n) \frac{(x+y)^{n+1}}{(n+1)!} \left(\frac{x+y}{n+2} + 4 + \right. \\ & \quad \left. + \frac{(x+y)^2}{(n+2)(n+3)} + 4 \frac{x+y}{n+2} \right) \leq 2^n (3ML^{n+1}K^{n+1}) \frac{(x+y)^{n+1}}{(n+1)!}. \end{aligned} \right.$$

In the same way we obtain and the required inequality for $|\partial u_{n+2}/\partial y - \partial u_{n+1}/\partial y|$ and all these inequalities are valid in the domain $0 \leq x \leq A$, $0 \leq y \leq A$.

Now, is clear that the series

$$(11) \quad \sum_{n=0}^{\infty} (u_{n+1}(x, y) - u_n(x, y)), \quad \sum_{n=0}^{\infty} \left(\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x} \right),$$

$$\sum_{n=0}^{\infty} \left(\frac{\partial u_{n+1}}{\partial y} - \frac{\partial u_n}{\partial y} \right)$$

are uniformly convergentes for $0 \leq x \leq A$, $0 \leq y \leq A$ i. e. the sequences

$$(12) \quad \{u_n(x, y)\}, \quad \left\{ \frac{\partial u_n}{\partial x} \right\}, \quad \left\{ \frac{\partial u_n}{\partial y} \right\}$$

are uniformly convergentes in the same domain. Passing to the limit in the recurrence relation (2), we conclude that $u = \lim_{n \rightarrow \infty} u_n$ is a solution of (1), having continuous partial derivatives of the first order for $x \geq 0$, $y \geq 0$ because $A > 0$ was arbitrary. Using a standard argument (see [1, pp. 126]), we can prove that the above solution u is unique.

2. It is well known that the hyperbolic equation

$$(13) \quad \begin{cases} \frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \\ u(x, 0) = \varphi(x) \\ u(0, y) = \psi(y) \end{cases}$$

where $f = f(x, y, u, v, w)$ is a continuous function for $x, y \geq 0$, $u, v, w \in R$ and φ, ψ belong to the class C^1 , $\varphi(0) = \psi(0)$ is equivalent with the following equation of the type (1)

$$(14) \quad u(x, y) = \varphi(x) + \psi(y) - \psi(0) + \iint_{00}^{xy} f\left(\xi, \eta, u(\xi, \eta), \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) d\xi d\eta.$$

Definition. We say that a function $f = f(x, y, u, v, w)$ defined on $\Delta = \{(x, y, u, v, w); x \geq 0, y \geq 0, u, v, w \in R\}$ satisfies the $(\mathcal{E}_1)$ condition if for every $A > 0$ there is a constant $L = L(A)$ such that if $0 \leq x \leq A$, $0 \leq y \leq A$ it follows

$$|f(x, y, u, v, w) - f(x, y, \bar{u}, \bar{v}, \bar{w})| \leq L(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|), \quad \forall u, v, w, \bar{u}, \bar{v}, \bar{w} \in R.$$

As a consequence of Theorem 1 we have

COROLLARY 1. Assume that the function $f = f(x, y, u, v, w)$ defined on Δ is continuous and satisfies the $(\mathcal{E}_1)$ condition. Assume also that φ and ψ belong to the class C^1 , $\varphi(0) = \psi(0)$.

Then the equation (13) has a unique solution $u = u(x, y)$ in the class C^1 .

Remark 1. If $f = a(x, y) + b(x, y)u + c(x, y)v + d(x, y)w$, where the functions a, b, c, d are continuous for $x, y \geq 0$, the conditions assumed in the preceding corollary are obviously satisfied and so we obtain the well known result concerning the existence and the uniqueness of the solution for the linear hyperbolic equation.

Remark 2. Turning back to the equation (1), we observe that if the function F is not dependent of the last two variables i. e. $F = F(x, y, \xi, \eta, u)$, we obtain the classical Volterra equation in which the unknown function u depends of two variables x and y .

We note also that the technique used in the proof of the Theorem 1 may be adapted for the case when in the right part of the equation (1) appear the derivatives of u of higher order and in the case when u depends of n variables.

Finally we give an other application of the Theorem 1, namely

COROLLARY 2. Let us consider the equation

$$(15) \quad \begin{cases} \frac{\partial^2 u}{\partial x \partial y} f = \left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) + \iint_{00}^{xy} g\left(x, y, \xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) d\xi d\eta, \\ u(x, 0) = \varphi(x), \quad u(0, y) = \psi(y) \end{cases}$$

where

(i) $f = f(x, y, u, v, w)$ is defined, continuous and satisfies the $(\mathcal{E}_1)$ condition in Δ .

(ii) $g = g(x, y, \xi, \eta, u, v, w)$ is defined and continuous in D and in addition the function G given by

$$G(x, y, \xi, \eta, u, v, w) = \iint_{\xi, \eta}^{x, y} g(\sigma, \tau, \xi, \eta, u, v, w) d\sigma d\tau,$$

satisfies the $(\mathcal{L})$ condition in D .

(iii) φ, ψ belong to the class C^1 and $\varphi(0) = \psi(0)$.

Then the equation (15) has a unique solution $u = u(x, y)$ in the class C^1 .

Proof. It suffices to observe that (15) is equivalent with an equation of the type (1), namely

$$(16) \quad u(x, y) = \varphi(x) + \psi(y) - \psi(0) + \iint_{0,0}^{x,y} H\left(x, y, \xi, \eta, u(\xi, \eta), \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) d\xi d\eta,$$

where

$$H(x, y, \xi, \eta, u, v, w) = f(\xi, \eta, u, v, w) + G(x, y, \xi, \eta, u, v, w)$$

satisfies the conditions required in Theorem 1.

Remark 3. If the function g is linear in u, v, w i.e. $g(x, y, \xi, \eta, u, v, w) = a + bu + cv + dw$, where the coefficients a, b, c, d are continuous functions of x, y, ξ, η the conditions assumed in the Corollary 2 are obviously satisfied.

Remark 4. Analogous equations with (1), but under other kind of conditions were considered in many papers, for example in [2], [3].

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O ECUAȚIE FUNCȚIONALĂ DE TIP VOLTERRA

(Rezumat)

Se rezolvă ecuația (1) prin metoda aproximațiilor succesive. Ecuația conține drept cazuri particulare ecuația clasică de tip Volterra, precum și o ecuație echivalentă cu (13). De asemenea, ecuația integro-diferențială (15) se reduce tot la o ecuație de tipul (1). Procedul folosit aici poate fi extins la cazul când funcția necunoscută u depinde de n variabile $x_1, x_2, \dots, x_n$ și când în membrul doi al ecuației (1) apar derivatele de ordin superior ale lui u .

ZUFÄLLIGE POTENZREIHEN MIT MULTIPLIKATIV ABHÄNGIGEN KOEFFIZIENTEN

VON
HARRO WALK

1. Einleitung

Häufigkeitsaussagen über Potenzreihen vom wahrscheinlichkeitstheoretischen Standpunkt aus wurden wohl zum ersten Male von Steinhaus [12] sowie von Paley und Zygmund [8] bewiesen. Die Autoren behandelten vor allem das lokale Verhalten am Rande des Konvergenzkreises und legten dabei zufällige Potenzreihen zugrunde, deren Koeffizienten mit Hilfe der sog. Steinhaus- und Rademacher-Funktionen definiert sind. Dabei ergab sich, daß hinsichtlich verschiedener Randeigenschaften entweder mit Wahrscheinlichkeit Eins überall ein singuläres oder mit Wahrscheinlichkeit Eins überall ein reguläres Verhalten vorliegt. Diese Resultate wurden — im Zusammenhang mit weiteren Randeigenschaften — u. a. von Ryll-Nardzewski [11], Arnold [2], Hinderer [4], Kahane [5] und dem Verfasser [13] für zufällige Potenzreihen mit unabhängigen in geeigneter Weise zentrierten Koeffizienten verallgemeinert.

In der vorliegenden Arbeit wird hierzu für einige dieser Randeigenschaften (Konvergenzverhalten, Charakteristik mit Verallgemeinerung, Fläche von Bildern, Länge von Bild-Kurvenstücken) gezeigt, daß die Voraussetzung der Unabhängigkeit der — am Erwartungswert zentrierten — Koeffizienten abgeschwächt werden kann zu gewissen Multiplikativitätsbedingungen (s. Abschnitt 2), falls die Koeffizienten einer Beschränkung ihrer Größenordnung genügen. Das mit Wahrscheinlichkeit Eins singuläre oder reguläre Verhalten der zufälligen Potenzreihen läßt sich dabei durch das Verhalten von Reihen charakterisieren, die mit geeigneten Erwartungswerten gebildet sind.

2. Vorbemerkungen

Zu einer Folge $\{a_n; n = 0, 1, \dots\}$ komplexwertiger Zufallsvariablen über einem Wahrscheinlichkeitsraum $(\Omega, \mathfrak{A}, P)$ werden Potenzreihen der

Form $\sum_{n=0}^{\infty} a_n(\omega) z^n$ mit $\omega \in \Omega$ betrachtet. Die Folge $\{a_n\}$ selbst wird als eine zufällige Potenzreihe mit Koeffizienten a_n bezeichnet.

Der Begriff eines stark multiplikativen Systems wurde von Alexits ([1], Ch. III, 2) eingeführt und von mehreren Autoren in verschiedenen Varianten mit schwächeren Bedingungen verwendet (s. z. B. Révész [10]). Im folgenden werden zufällige Potenzreihen $\{a_n\}$ auf $(\Omega, \mathfrak{F}, P)$ behandelt, die den nachstehenden Integrierbarkeits- und Multiplikativitätsbedingungen (vgl. [14]) genügen:

- (1) $E |a_n|^4 < \infty$, $E a_n = 0$ für alle n ;
- (2) $E(a_{n_1} a_{n_2} \overline{a_{n_3}} \overline{a_{n_4}}) = 0$, wenn mindestens 3 Indizes verschieden sind;
- (3) $E(|a_{n_1}|^2 |a_{n_2}|^2) = E |a_{n_1}|^2 E |a_{n_2}|^2$ für alle $n_1 \neq n_2$.

Die Bedingung $E a_n = 0$ in (1) tritt in den Beweisen der späteren Resultate nicht auf und ist nur der Übersichtlichkeit halber aufgeführt.

Es werden die Bezeichnungen $\mathbf{N}$ und $\mathbf{R}_+$ für die Menge der natürlichen Zahlen bzw. die Menge der nichtnegativen reellen Zahlen verwendet; f. s. bedeutet fast sicher. Meßbarkeitsfragen sind bei Hinderer [4] behandelt.

3. Unbeschränktheit der Charakteristik und damit zusammenhängende Eigenschaften

Der nachstehende Satz 1 in Verbindung mit Bemerkung 1c besagt insbesondere, daß für eine zufällige Potenzreihe $\{a_n\}$, deren Koeffizienten gleichmäßig beschränkt sind und den obigen Multiplikativitätsbedingungen genügen, $\sum a_n z^n$ entweder f. s. in jedem Sektor des Einheitskreises $|z| < 1$ von unbeschränkter Charakteristik ist (genau dann ist $\sum E |a_n|^2 = \infty$ oder auch f. s. $\sum |a_n|^2 = \infty$) oder f. s. in $|z| < 1$ von beschränkter Charakteristik ist (genau dann ist $\sum E |a_n|^2 < \infty$ oder auch f. s. $\sum |a_n|^2 < \infty$, also $\sum a_n z^n$ sogar f. s. aus der Klasse H_2); entsprechendes gilt bis auf L -Nullmengen für Divergenz und Konvergenz auf dem Rand des Einheitskreises.

Satz 1. *Geben seien eine zufällige Potenzreihe $\{a_n\}$ auf $(\Omega, \mathfrak{F}, P)$ mit (1), (2), (3) und eine stetige Funktion $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ mit $h(s) \rightarrow \infty$ ($s \rightarrow \infty$).*

Gilt

$$\sum E |a_n|^2 = \infty, \quad \lim_n \overline{(E |a_n|)^{\frac{1}{n}}} \leq 1 \quad \text{und}$$

$$(4) \quad \text{ess sup}_{\omega \in \Omega} |a_N(\omega)|^2 = o\left(\sum_{n=0}^N E |a_n|^2\right) \quad (N \rightarrow \infty)$$

so ist P -f. s.

$$(5) \quad \sup_{0 < \rho < 1} \int_{\alpha}^{\beta} h(|\sum a_n \rho^n e^{in\varphi}|) d\varphi = \infty \text{ für alle } \alpha, \beta \text{ mit } 0 < \beta - \alpha \leq 2\pi,$$

$$(6) \quad \sum |a_n|^2 = \infty,$$

$$(7) \quad \sum a_n e^{in\varphi} \text{ für } L\text{-fast alle } \varphi \in [0, 2\pi) \text{ divergent.}$$

Bemerkung 1. a) Durch $h = \log^+$ in Satz 1 wird der Fall der unbeschränkten Charakteristik erfaßt. Die Bedingung $\lim (E|a_n|)^{\frac{1}{n}} \leq 1$ ist hier jeweils äquivalent mit

$$\sum E|a_n|^2 \rho^{2n} < \infty \quad (0 \leq \rho < 1)$$

und mit

$$P[\sum a_n z^n \text{ konvergent in } |z| < 1] = 1.$$

Dies ergibt sich mit der Cauchy-schwarzschen Ungleichung und dem sich auf (6) beziehenden Teil des Satzes 1.

b) Für eine zufällige Potenzreihe $\{a_n\}$ über $(\Omega, \mathfrak{A}, P)$ gelten P -f. s. (6) und (7) auch unter den schwächeren Voraussetzungen (1), (3) mit $E|a_n| > 0$ für mindestens ein n und der Lindeberg-Bedingung

$$\left(\sum_{n=0}^N E|a_n|^2 \right)^{-1} \sum_{n=0}^N \int_{C_{N,\varepsilon,n}} |a_n|^2 dp \rightarrow 0 \quad (N \rightarrow \infty) \quad (\varepsilon > 0)$$

mit

$$C_{N,\varepsilon,n} := \left[|a_n| \geq \varepsilon \left(\sum_{k=0}^N E|a_k|^2 \right)^{\frac{1}{2}} \right].$$

Unter den angegebenen Voraussetzungen ist außerdem für jedes $\varphi \in [0, 2\pi)$ P -f. s. $\sum a_n e^{in\varphi}$ divergent (vgl. [14] und M ó r i c z [7]).

c) Für eine zufällige Potenzreihe $\{a_n\}$ mit $\sum E|a_n|^2 < \infty$, aber nicht notwendig (1), (2), (3), und eine stetige Funktion $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ mit $h(s) = O(s^2)$ ($s \rightarrow \infty$) ist f. s.

$$\sup_{0 < \rho < 1} \int_0^{2\pi} h(|\sum a_n \rho^n e^{in\varphi}|) d\varphi < \infty, \quad \sum |a_n|^2 < \infty,$$

$\sum a_n e^{in\varphi}$ für L -fast alle $\varphi \in [0, 2\pi)$ konvergent, wie sich unmittelbar mit bekannten Resultaten, insbesondere einem Ergebnis von Carleson [3], ergibt.

d) Wie Gegenbeispiele zeigen, darf man in Satz 1 die Beschränktheitsbedingung (4) dicht weglassen; auch darf die Voraussetzung (2) nicht ersetzt werden durch die Bedingung, daß die Koeffizienten a_n orthogonal im üblichen Sinne sind. Für eine zufällige Potenzreihe $\{a_n\}$ mit (1), (2), (3), gleichmäßig beschränkten Koeffizienten und f. s. Konvergenzradius Eins braucht bezüglich der gewöhnlichen Singularität kein gleichartiges Randverhalten vorzuliegen, es kann nämlich $\sum a_n z^n$ f. s. über $|z| = 1$ hinaus fortsetzbar sein.

Beweis von Satz 1 und Bemerkung 1 b. Der Hauptteil des Beweises besteht im Nachweis der ersten Implikation des Satzes 1, also im Schluß auf die P -f. s. Gültigkeit von (5). Dieser soll indirekt geführt werden. Ohne Beschränkung der Allgemeinheit kann man h als monoton wachsend und $h(x+y) \leq h(x) + h(y)$ ($x, y \in \mathbf{R}^+$) voraussetzen. Die Annahme beim indirekten Beweis bedeutet, daß ein Paar (α, β) mit $0 < \beta - \alpha \leq 2\pi$ so existiert, daß für

$$\Omega_0 := \left\{ \omega \in \Omega : \sup_{0 < \rho < 1} \int_{\alpha}^{\beta} h\left(\left|\sum a_n \rho^n e^{in\varphi}\right|\right) d\varphi < \infty \right\} \in \mathfrak{F}$$

die Beziehung

$$\delta := P(\Omega_0) > 0$$

gilt.

Ausgehend von einem einzelnen $\omega \in \Omega_0$ kann man leicht zeigen: Es existieren ein $\Omega' \in \mathfrak{F}$ mit $\Omega' \subset \Omega_0$, $P(\Omega') \geq \frac{\delta}{2}$ sowie eine Folge $\{\rho_v\}$ mit $0 < \rho_v \uparrow 1$, eine Indexfolge $\{N_v\}$ mit $N_v \uparrow \infty$ und eine Zahl $K < \infty$ derart, daß in Ω' jedes a_n beschränkt ist und

$$(8) \quad \int_{\alpha}^{\beta} h\left(\left|\sum_{n=0}^{N_v} a_n(\omega) \rho_v^n e^{in\varphi}\right|\right) d\varphi \leq K \quad (\omega \in \Omega', v \in \mathbf{N})$$

gilt.

Mit

$$A_v := \sum_{n=0}^{N_v} E |a_n|^2 \rho_v^{2n} \quad \text{und} \quad q_v := P\left[\left|\sum_{n=0}^{N_v} |a_n|^2 \rho_v^{2n} - A_v\right| \geq \frac{1}{4} A_v\right]$$

strebt

$$q_v \rightarrow 0 \quad (v \rightarrow \infty);$$

denn mit der Tschebyscheffschen Ungleichung ergibt sich (vgl. [14], S. 55)

$$q_v \leq \frac{16}{A_v} \max_{n=0, \dots, N_v} \left\{ \operatorname{ess\,sup}_{\omega \in \Omega} |a_n(\omega)|^2 \rho_v^{2n} \right\}$$

und daraus wegen $A_v \uparrow \infty$ ($v \rightarrow \infty$) und (4) die angegebene Beziehung.

Ohne Beschränkung der Allgemeinheit kann

$$\sum_{v=1}^{\infty} q_v < \frac{\delta}{4}$$

angenommen werden. Dann existiert ein $\Omega'' \in \mathfrak{K}$ mit $P(\Omega'') \geq 1 - \frac{\delta}{4}$ derart, daß

$$(9) \quad \frac{3}{4} A_v \leq \sum_{n=0}^{N_v} |a_n(\omega)|^2 \rho_v^{2n} \leq \frac{5}{4} A_v \quad (\omega \in \Omega'', v \in \mathbb{N})$$

gilt.

Es wird nun $\Omega^* := \Omega' \cap \Omega''$ gesetzt. Es ist $P(\Omega^*) \geq \frac{\delta}{4} > 0$. Die Beziehungen (8), (9) gelten für $\omega \in \Omega^*$, $v \in \mathbb{N}$.

Mit Hilfe einer Überlegung von Zygmund ([15], S. 203 f.) erhält man wegen (2) und (3) — ausgehend von einem einzelnen $\varphi \in [\alpha, \beta]$; vgl. [14, S. 54] —: Es existieren eine Borelsche Teilmenge Φ von $[\alpha, \beta]$ mit Lebesgue-Borel-Maß $|\Phi| > 0$ und ein Index $M \in \mathbb{N}$ so, daß mit $A_v^* :=$

$$(10) \quad = \sum_{n=M}^{N_v} E |a_n|^2 \rho_v^{2n} \text{ gilt} \\ \left| \int_{\Omega^*} \left(\sum_{\substack{j,k \in \{M, \dots, N_v\} \\ j \neq k}} a_j \rho_v^j e^{ij\varphi} \overline{a_k \rho_v^k e^{ik\varphi}} \right) dP \right| \leq \frac{1}{4} P(\Omega^*) A_v^*, \quad (\varphi \in \Phi, v \in \mathbb{N}).$$

Aus (8) folgt die Existenz eines $K^* < \infty$ mit

$$(11) \quad \int_{\alpha}^{\beta} h \left(\left| \sum_{n=M}^{N_v} a_n(\omega) \rho_v^n e^{in\varphi} \right| \right) d\varphi \leq K^* \quad (\omega \in \Omega^*, v \in \mathbb{N}).$$

Aus (9) folgt die Existenz eines $v^* \in \mathbb{N}$ mit

$$(12) \quad \frac{1}{2} A_v^* \leq \sum_{n=M}^{N_v} |a_n(\omega)|^2 \rho_v^{2n} \leq \frac{3}{2} A_v^* \quad (\omega \in \Omega^*, v \geq v^*).$$

Die erste Ungleichung in (12) und die Beziehung (10) implizieren

$$\int_{\Omega^*} \left| \sum_{n=M}^{N_v} a_n \rho_v^n e^{in\varphi} \right|^2 dP \geq \frac{1}{4} P(\Omega^*) A_v^* \quad (\varphi \in \Phi, v \geq v^*);$$

die zweite Ungleichung in (12) impliziert wegen

$$\left| \sum_{n=M}^{N_v} a_n \varphi_v^n e^{in\varphi} \right|^4 \leq 2 \left(\sum_{n=M}^{N_v} |a_n|^2 \varphi_v^{2n} \right)^2 + 2 \left(\sum_{\substack{j,k \in \{M, \dots, N_v\} \\ j \neq k}} a_j \varphi_v^j e^{ij\varphi} \overline{a_k \varphi_v^k e^{ik\varphi}} \right)^2$$

zusammen mit (2), (3) die Abschätzung

$$\int_{\Omega^*} \left| \sum_{n=M}^{N_v} a_n \varphi_v^n e^{in\varphi} \right|^4 dP \leq \frac{13}{2} A_v^{*2} \quad (\varphi \in \Phi, v \geq v^*) .$$

Jetzt erhält man gemäß Paley und Zygmund (s. [15, S. 216f.]) für

$$\Omega_{v\varphi}^* := \left\{ \omega \in \Omega^* : \left| \sum_{n=M}^{N_v} a_n(\omega) \varphi_v^n e^{in\varphi} \right|^2 \geq \frac{1}{8} A_v^* \right\}$$

die Beziehung

$$P(\Omega_{v\varphi}^*) \geq \frac{1}{416} P(\Omega^*)^2 \quad (\varphi \in \Phi, v \geq v^*)$$

und daraus

$$\begin{aligned} & \int_{\Omega^*} \int_{\alpha}^{\beta} h \left(\left| \sum_{n=M}^{N_v} a_n(\omega) \varphi_v^n e^{in\varphi} \right| \right) P(d\omega) d\varphi \geq \\ & \geq \frac{1}{416} P(\Omega^*)^2 h \left(\sqrt{\frac{1}{8} A_v^*} \right) \quad |\Phi| \rightarrow \infty \quad (v \rightarrow \infty). \end{aligned}$$

Andererseits ergibt sich aus (11) die Beziehung

$$\int_{\Omega^*} \int_{\alpha}^{\beta} h \left(\left| \sum_{n=M}^{N_v} a_n(\omega) \varphi_v^n e^{in\varphi} \right| \right) d\varphi P(d\omega) \leq K^* P(\Omega^*) \quad (v \in \mathbb{N}).$$

Nach Vertauschung der Integrationsreihenfolge ergibt sich hieraus ein Widerspruch zum vorhergehenden Ergebnis. Damit ist die P -f. s. Gültigkeit von (5) nachgewiesen.

Se konvergiert

$$\left(E \sum_{n=0}^N |a_n|^2 \right)^{-1} \sum_{n=0}^N |a_n|^2 \rightarrow 1 \quad (N \rightarrow \infty) \text{ in Wahrscheinlichkeit,}$$

wie sich aus dem obigen Beweis (mit der Tschebyscheffschen Ungleichung) und für die in Bemerkung 1 b angegebenen schwächeren Voraussetzungen aus Überlegungen bei Komlóš [6] und Révész [10] unmittelbar ergibt. Daraus erhält man die sich auf (6) beziehende Aussage in Satz

I und in Bemerkung 1b sofort und die sich auf (7) beziehende Aussage ähnlich wie im obigen Beweis, nur etwas einfacher, wobei es genügt, sich auf ein beliebiges festes $\varphi \in [0, 2\pi)$, etwa $\varphi = 0$, zu beschränken.

4. Weitere Randeigenschaften

Die in diesem Abschnitt angegebenen Resultate beziehen sich auf die durch zufällige Potenzreihen vermittelten Bilder von Sektoren des Einheitskreises und von konzentrischen Kreisbögen im Einheitskreis. Im ersten Falle verläuft der Beweis wiederum ähnlich wie bei Satz 1 und wird deshalb nicht durchgeführt; im zweiten Falle erhält man das Resultat leicht aus Satz 1 unter Verwendung der einmal abgeleiteten Potenzreihe. Die zugehörigen funktionentheoretischen Hilfsmittel finden sich in [9] und [4].

Satz 2. Gegeben sei eine zufällige Potenzreihe $\{a_n\}$ auf $(\Omega, \mathfrak{K}, P)$ mit (1), (2), (3).

a) Gilt $\sum nE|a_n|^2 = \infty$, $\lim_{n \rightarrow \infty} (E|a_n|)^{\frac{1}{n}} \leq 1$ und

$$\operatorname{ess\,sup}_{\omega \in \Omega} N|a_N(\omega)|^2 = o\left(\sum_{n=0}^N nE|a_n|^2\right) \quad (N \rightarrow \infty),$$

so bildet P -f. s. $\sum a_n z^n$ jeden offenen Sektor der Kreisscheibe $|z| < 1$ auf ein Gebiet unendlichen Flächeninhalts ab.

b) Gilt $\sum n^2 E|a_n|^2 = \infty$, $\lim_{n \rightarrow \infty} (E|a_n|)^{\frac{1}{n}} \leq 1$ und

$$\operatorname{ess\,sup}_{\omega \in \Omega} N^2|a_N(\omega)|^2 = o\left(\sum_{n=0}^N n^2 E|a_n|^2\right) \quad (N \rightarrow \infty),$$

so bildet $\sum a_n z^n$ für jedes feste Paar (α, β) mit $0 < \beta - \alpha \leq 2\pi$ die Kreisbögen $\{\rho e^{i\varphi} : \alpha \leq \varphi \leq \beta\}$ ($0 < \rho < 1$) auf Kurvenstücke ab, deren Längen für $\rho \rightarrow 1$ nicht beschränkt bleiben.

Aus Satz 2 ergibt sich in Verbindung mit Satz 1 das folgende Resultat. Genügen die Koeffizienten a_n einer zufälligen Potenzreihe den obigen Multiplikativitätsbedingungen und ist die Folge $\{\sqrt[n]{n} a_n\}$ gleichmäßig beschränkt, so bildet $\sum a_n z^n$ entweder f. s. jeden offenen Sektor der Kreisscheibe $|z| < 1$ auf ein Gebiet unendlichen Flächeninhalts ab (genau dann ist $\sum nE|a_n|^2 = \infty$ oder auch f. s. $\sum n|a_n|^2 = \infty$) oder f. s. die Kreisscheibe $|z| < 1$ auf ein Gebiet endlichen Flächeninhalts (genau dann ist $\sum nE|a_n|^2 < \infty$ oder auch f. s. $\sum n|a_n|^2 < \infty$). Ein ähnliches Resultat ergibt sich für den Fall der Kreisbögen.

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SERII ALEATOARE DE PUTERI CU COEFICIENȚI MULTIPLICATIV
INDEPENDENȚI

(Rezumat)

Se demonstrează că comportarea singulară sau regulată a seriilor aleatoare de puteri poate fi caracterizată cu probabilitatea egală cu unitatea prin comportarea unor serii construite cu ajutorul valorilor probabile.

CRITERIU DE VERIFICARE A IPOTEZELOR STATISTICE PARAMETRICE ÎN TEORIA SIGURANȚEI

DE

GH. FLOREA

1. Legea exponențială este foarte populară în teoria siguranței, ceea ce se explică prin faptul că este foarte naturală din punct de vedere fizic, simplă și comod de folosit.

Să presupunem deci, că timpul de funcționare fără întrerupere a unui element are funcția de repartiție

$$F(t) = 1 - e^{-\lambda t},$$

unde $\lambda > 0$ este valoarea necunoscută a parametrului. În teoria siguranței, cel mai mare interes îl prezintă verificarea ipotezelor statistice compuse de forma :

$$H_0 = \{\lambda \leq \lambda_0\}, \quad H_1 = \{\lambda > \lambda_0\},$$

unde λ_0 este o valoare fixată a parametrului. Dacă ne amintim că funcția pericol de defectare definită prin :

$$r(t) = \frac{f(t)}{\bar{F}(t)}, \quad (f(t) = F'(t), \bar{F}(t) = 1 - F(t))$$

în cazul legii exponențiale este egală cu λ , atunci, putem considera că valoarea λ_0 este aleasă astfel încît pentru acele valori ale lui λ pentru care $\lambda \leq \lambda_0$ elementul se poate considera sigur, iar pentru valorile $\lambda > \lambda_0$, re-sigur.

2. Pentru construirea criteriului de verificare a ipotezelor de mai sus ne vom baza pe teorema lui Neyman - Pearson. Să presupunem că se experimentează N elemente, elementele defectate înlocuindu-se cu altele identice. În plus, să admitem că observațiile asupra defectărilor se efectuează în mod continuu și că experiența se termină în momentul apariției

cei de a r -a defecțiuni ($r \leq N$). Dacă vom nota prin $t_1, t_2, \dots, t_r$ momentele apariției, respectiv a primei defecțiuni, a celei de a doua defecțiuni și așa mai departe, a celei de a r -a defecțiuni, iar prin $S(t^*)$ — suma timpilor de funcționare a tuturor elementelor până la momentul t^* , atunci $t^* = t_r$, iar $S(t^*) = Nt_r$.

Spațiul $X = \{(x_1, x_2, \dots, x_N)\}$ al rezultatelor, este alcătuit din șirul momentelor de defecție $t_i \leq t_r$, respectiv momentul terminării experienței t_r , adică

$$x_i = t_i \quad \text{pentru} \quad i \leq r$$

și

$$x_i = t_r \quad \text{pentru} \quad r < i \leq N.$$

Având în vedere că timpul de funcționare al elementelor s-a presupus repartizat exponențial de parametru λ , rezultă că defecțiunile fiecărui element formează un flux Poisson de parametru λ , de unde observația că suprapunerea celor N fluxuri Poisson independente, corespunzătoare celor N elemente, formează un flux Poisson de parametru $N\lambda$ și deci, intervalele de timp dintre două defecțiuni succesive sînt repartizate exponențial de parametru $N\lambda$. Ținînd cont de această ultimă observație va rezulta că densitatea de probabilitate a evenimentului ce constă în apariția primei defecțiuni în intervalul $(t_1, t_1 + dt_1)$, a celei de a doua defecțiuni în intervalul $(t_2, t_2 + dt_2)$ condiționată de faptul că prima defecțiune a apărut în intervalul $(t_1, t_1 + dt_1)$, și așa mai departe, a celei de a r -a defecțiuni în intervalul $(t_r, t_r + dt_r)$ condiționată de faptul că celelalte defecțiuni au apărut în intervalele $(t_1, t_1 + dt_1)$, $(t_2, t_2 + dt_2), \dots, (t_{r-1}, t_{r-1} + dt_{r-1})$ va fi egală cu:

$$(N\lambda) e^{-N\lambda t_1} dt_1 (N\lambda) e^{-N\lambda(t_2 - t_1)} dt_2 \dots (N\lambda) e^{-N\lambda(t_r - t_{r-1})} dt_r = \\ = (N\lambda)^r e^{-N\lambda t_r} dt_1 dt_2 \dots dt_r = (N\lambda)^r e^{-\lambda S(t_r)} dt_1 dt_2 \dots dt_r,$$

de unde găsim că funcția de verosimilitate este egală cu:

$$p(t_1, t_2, \dots, t_r | \lambda) = (N\lambda)^r e^{-\lambda S(t_r)}.$$

Să mai facem observația că singura statistică suficientă este $S(t_r)$.

Potrivit teoremei lui Neyman-Pearson, pentru două ipoteze parametrice altele native, există un cel mai puternic test, absolut corect, care oferă în același timp și modul de determinare a celei mai bune regiuni critice. Fie $\lambda_1 > \lambda_0$ o a doua valoare a parametrului λ . Atunci regiunea critică W_α de prag de semnificație cel mult α , va fi aleasă dintre mulțimile:

$$W_c = \{x | p(x | \lambda_1) \geq cp(x | \lambda_0)\}.$$

Cum

$$p(x | \lambda_1) = (N\lambda_1)^r e^{-\lambda_1 S(t_r)} \quad \text{și} \quad p(x | \lambda_0) = (N\lambda_0)^r e^{-\lambda_0 S(t_r)},$$

atunci

$$\frac{p(x | \lambda_1)}{p(x | \lambda_0)} = \left(\frac{\lambda_1}{\lambda_0} \right)^r e^{-(\lambda_1 - \lambda_0) S(t_r)},$$

de unde observația că :

$$W_c = \left\{ x \mid r \ln \frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0) S(t_r) \geq c \right\}.$$

Această ultimă relație ne permite să afirmăm că regiunea critică W_α , de prag de semnificație cel mult α , va fi aleasă dintre mulțimile

$$W_S = \{ x \mid S(t_r) < S \}.$$

Cum t_r poate fi scris ca o sumă de r v. a. independente repartizate exponențial de parametru $N\lambda$, adică

$$t_r = t_1 + (t_2 - t_1) + \dots + (t_r - t_{r-1}),$$

vom putea deduce că

$$\begin{aligned} P \{ S(t_r) < S \mid \lambda \} &= P \{ Nt_r < S \mid \lambda \} = P \left\{ t_r < + \frac{S}{N} \mid \lambda \right\} = \int_0^{\frac{\lambda S}{N}} \frac{t^{r-1}}{(r-1)!} e^{-t} dt = \\ &= 1 - \left(1 + \frac{\lambda S}{1!} + \frac{(\lambda S)^2}{2!} + \dots + \frac{(\lambda S)^{r-1}}{(r-1)!} \right) e^{-\lambda S} = 1 - L_{r-1}(\lambda S), \end{aligned}$$

unde funcția $L_c(x)$ este monoton necrescătoare de x și monoton crescătoare de c . Atunci regiunea critică de prag de semnificație cel mult α va fi cea determinată de cel mai mare S , pe care-l vom nota S_α , pentru care

$$P \{ S(t_r) < S_\alpha \mid \lambda_0 \} = 1 - L_{r-1}(\lambda_0 S),$$

de unde se obține

$$S_\alpha = \frac{\Delta_{1-\alpha}(r-1)}{\lambda_0}.$$

Astfel, pentru α și r dați, λ_0 ales, din tabele se determină $\Delta_{1-\alpha}(r-1)$, și folosind formula precedentă se determină S_α , care se compară cu $S(t_r)$, astfel încît, dacă $S(t_r) < S_\alpha$ atunci ipoteza H_0 se va respinge și deci se va accepta ipoteza H_1 , iar dacă $S(t_r) > S_\alpha$, se va accepta ipoteza H_0 .

3. Funcția putere a acestui criteriu este :

$$P \{ x \mid S(t_r) < S_\alpha ; \lambda \} = 1 - L_{r-1} \left(\Delta_{1-\alpha}(r-1) \frac{\lambda}{\lambda_0} \right).$$

Ținînd cont de monotonia funcției $L_c(x)$ ca funcție de x și de alegerea lui S_α putem afirma că testul astfel construit este uniform absolut corect în raport cu mulțimea $\omega = (0, S_\alpha)$. Să mai observăm că eroarea de genul doi



este egală cu $L_{r-1}(\lambda S)$, care devine minimă pentru $S = S_\alpha$ în raport cu toți a ei S pentru care pragul de semnificație este cel mult α , și deci criteriul astfel construit este cel mai puternic.

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CRITERION FOR TESTING PARAMETRICAL STATISTICAL HYPOTHESES IN THE THEORY OF SURENESS

(Summary)

In this paper, the strongest, absolutely correct criterion for testing statistical parametric hypotheses of the form:

$$H_0 = \{\lambda \leq \lambda_0\}, \quad H_1 = \{\lambda > \lambda_0\}$$

is obtained, where λ stands for the parameter of exponential repartition

$$F(t) = 1 - e^{-\lambda t}$$

of the elements continuously working time.

DISTRIBUTIONS DES ACCÉLÉRATIONS DÉTERMINÉES PAR LA MÉTHODE DES COORDONNÉES VECTORIELLES PLANE-AXIELLES

PAR

C. HUIU, N. IRIMICIUC, ELISABETA RUSU et M. STANCU

0. Introduction

Les auteurs de cette Note, conséquents à l'idée développée dans [6] et [7], appliquent maintenant la méthode des coordonnées vectorielles plane-axielles, exposée dans la suite des travaux [2] — [5], pour la détermination des accélérations des solides rigides en mouvement général.

Les notations utilisées sont les mêmes comme dans [6] et [7].

1. Les composantes plane-axielles des paramètres cinématiques de deuxième ordre du mouvement général du solide

Il faut considérer de nouveau le repère $Q_p N_1 N_2 N_3$ (fig. 1) formé par la projection $Q_p N_1$ de la ligne des neuds sur le plan de la représentation, par le parallèle $Q_p N_3$ à l'axe de la représentation et par l'axe $Q_p N_2$ qui forme avec les premières deux un trièdre triorthogonale droit. Les vecteurs unitaires de ces axes sont $\bar{n}_1 = \bar{n}$, $\bar{n}_2$ et $\bar{i}_3$.

Les paramètres cinématiques de deuxième ordre du mouvement général du solide auront les expressions

$$(1.1) \quad \bar{a}_Q = \bar{a}_{Q_{II}} + \bar{a}_{Q_{\Delta}}, \quad \bar{\varepsilon} = \bar{\varepsilon}_{II} + \bar{\varepsilon}_{\Delta}$$

où les composantes plane-axielles seront données par les relations

$$(1.2) \quad \begin{cases} \bar{a}_{Q\Pi} = (r_Q - r_Q \dot{\varphi}_Q^2) \bar{i}_{r_Q} + (2\dot{r}_Q \dot{\varphi}_Q + r_Q \ddot{\varphi}_Q) \bar{i}_{\varphi_Q}, \\ \bar{a}_{Q\Delta} = \ddot{z}_Q \bar{i}_3 \end{cases}$$

et

$$(1.3) \quad \begin{cases} \bar{\varepsilon}_{\Pi} = (\ddot{\theta}_n + \dot{\theta}_r \dot{\theta}_p \sin \theta_n) \bar{n}_1 + (\dot{\theta}_n \dot{\theta}_p - \ddot{\theta}_r \sin \theta_n - \dot{\theta}_r \dot{\theta}_p \cos \theta_n) \bar{n}_2, \\ \bar{\varepsilon}_{\Delta} = (\ddot{\theta}_p + \ddot{\theta}_r \cos \theta_n - \dot{\theta}_r \dot{\theta}_n \sin \theta_n) \bar{i}_3. \end{cases}$$

L'introduction des échelles

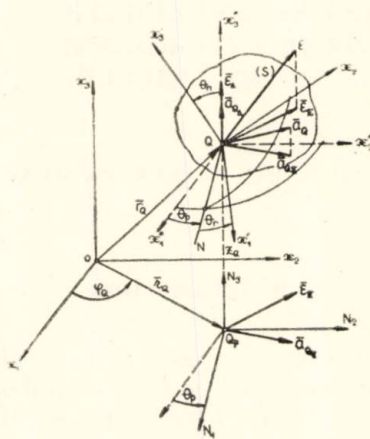


Fig. 1

$$(1.4) \quad \sigma_{\varepsilon} = \sigma_{\omega}^2 \cdot 1 \text{ cm} = \left(\frac{v_{\omega}}{q_{\omega}} \right)^2 \left[\frac{\text{s}^{-2}}{\text{cm}} \right]$$

pour la représentation des accélérations angulaires (σ_{ω} étant l'échelle des vitesses angulaires) et

$$(1.5) \quad \sigma_a = \sigma_L \sigma_{\omega}^2 (1 \text{ cm})^2 = \left(\frac{v_L v_{\omega}^2}{q_L q_{\omega}^2} \right) \left[\frac{\text{m/s}^2}{\text{cm}} \right]$$

pour la représentation des accélérations linéaires (σ_L étant l'échelle des distances), permettra d'écrire

$$(1.6) \quad \bar{a}_Q = \sigma_a (\bar{a}_{Q\Pi}^r + \bar{a}_{Q\Delta}^r), \quad \bar{\varepsilon} = \sigma_{\varepsilon} (\bar{\varepsilon}_{\Pi}^r + \bar{\varepsilon}_{\Delta}^r),$$

ù

$$(1.7) \quad \begin{cases} \bar{a}_{Q\Pi}^r = (\ddot{r}_Q^r - r_Q^r \dot{\varphi}_Q^r) \bar{i}_{rQ} + (2\dot{r}_Q^r \dot{\varphi}_Q^r + r_Q^r \ddot{\varphi}_Q^r) \bar{i}_{\varphi Q}, \\ \bar{a}_{Q\Delta}^r = \ddot{z}_Q^r \bar{i}_3 \end{cases}$$

sont les composantes plane-axielles du vecteur représentatif de l'accélération de translation et

$$(1.8) \quad \begin{cases} \bar{\varepsilon}_\Pi^r = (\ddot{\theta}_n^r + \dot{\theta}_r^r \dot{\theta}_p^r \sin \theta_n) \bar{n}_1 + (\dot{\theta}_n^r \dot{\theta}_p^r - \ddot{\theta}_r^r \sin \theta_n - \dot{\theta}_r^r \dot{\theta}_n^r \cos \theta_n) \bar{n}_2, \\ \bar{\varepsilon}_\Delta^r = (\ddot{\theta}_p^r + \ddot{\theta}_r^r \cos \theta_n - \dot{\theta}_r^r \dot{\theta}_n^r \sin \theta_n) \bar{i}_3 \end{cases}$$

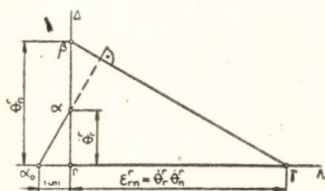


Fig. 2

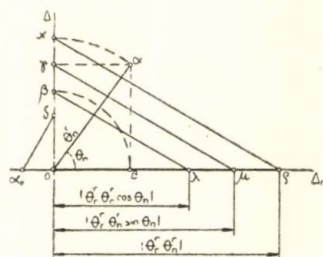


Fig. 3

sont les composantes plane-axielles du vecteur représentatif de l'accélération angulaire.

Les fig. 2 et 3 indiquent les constructions graphiques qui déterminent les divers termes de ces composantes.

2. Détermination de l'accélération d'un point arbitraire du solide en mouvement général

Les procédés graphiques élaborés pour la détermination des accélérations sont fondés sur les trois observations suivantes :

a) Tous les points du solide situés sur un parallèle à la ligne d'action du vecteur $\bar{\varepsilon}$ ont la même accélération de rotation, c'est-à-dire qu'on peut écrire (fig. 4) :

$$(2.1) \quad \bar{a}_{\text{rot}D_1} = \bar{\varepsilon} \times \bar{r}'_{D_1} = \bar{\varepsilon} \times \bar{r}'_B = \bar{a}_{\text{rot}B}.$$

b) L'accélération de rotation a le même module, la même direction et le sens contraire que le moment par rapport au pôle Q , du vecteur $\bar{\varepsilon}$ considéré appliqué en B :

$$(2.2) \quad \bar{a}_{\text{rot}} = \bar{a}_{\text{rot}_B} = \bar{\varepsilon} \times \bar{r}' = -\bar{r}' \times \bar{\varepsilon} = -\bar{M}_Q(\bar{\varepsilon}_B)$$

(la notation $\bar{\varepsilon}_B$ indique que le vecteur $\bar{\varepsilon}$ doit être considéré appliqué en B);

c) L'accélération axipète est la même, à un instant quelconque, pour tous les points du solide situés sur le même parallèle à la ligne d'action du vecteur $\bar{\omega}_B$:

$$(2.3) \quad \bar{a}_{ax_{B_1}} = \bar{\omega} \times (\bar{\omega} \times \bar{r}'_{B_1}) = \bar{\omega} \times (\bar{\omega} \times \bar{r}) = \bar{a}_{ax_B}.$$

Tout en concordance avec les observations précédentes on prendra le point $D_0(\bar{r}'_{D_0})$ (fig. 4), pour la détermination de l'accélération de rotation du point B et le point $B_0(\bar{r}'_0)$ pour la détermination de l'accélération axipète du ce point, ce qui permet d'écrire

$$(2.4) \quad \bar{a}_B = \bar{a}_Q - \bar{M}_Q(\bar{\varepsilon}_B) + \bar{\omega} \times (\bar{\omega} \times \bar{r}'_0) = \sigma_a[(\bar{a}_{Q_{\Pi}} + \bar{a}_{\text{rot}_{\Pi}} + \bar{a}_{ax}) + (\bar{a}_{Q_{\Delta}} + \bar{a}_{\text{rot}_{\Delta}} + \bar{a}_{ax_{\Delta}})].$$

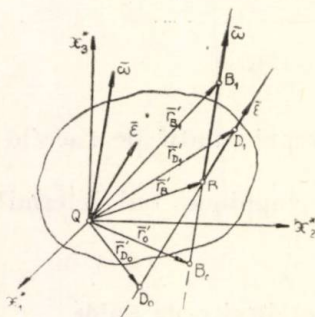


Fig. 4

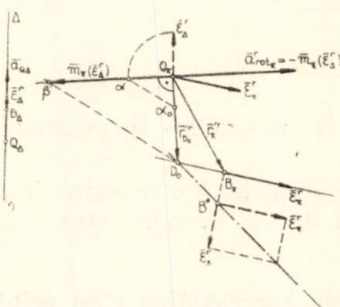


Fig. 5

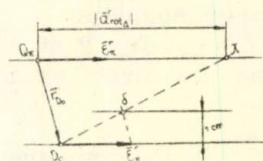


Fig. 6

Les composantes plane-axielles du vecteur représentatif de l'accélération de translation ont été déterminées dans la première partie de la note.

Les constructions graphiques pour la détermination du vecteur représentatif

$$(2.5) \quad \bar{a}_{\text{rot}_{\Pi}} = -\bar{m}_{\Pi}(\bar{\varepsilon}_{\Pi})$$

sont indiquées dans la fig. 5 et celles qui déterminent la composante $\bar{a}_{\text{rot}_{\Delta}}$, dans la fig. 6.

Enfin, dans la fig. 7 sont présentées les constructions graphiques qui déterminent les composantes plane-axielles du vecteur représentatif de l'accélération axipète, en vertu des relations

$$(2.6) \quad \begin{cases} \bar{a}_{ax\Pi}^r \cdot (1 \text{ cm})^2 = (\bar{\omega}_{\Pi}^r \bar{r}_0^r) \bar{\omega}_H^r - \omega^2 \bar{r}_0^r, \\ \bar{a}_{ax\Delta}^r \cdot (1 \text{ cm})^2 = (\bar{\omega}_{\Pi}^r \bar{r}_0^r) \bar{\omega}_{\Delta}^r. \end{cases}$$

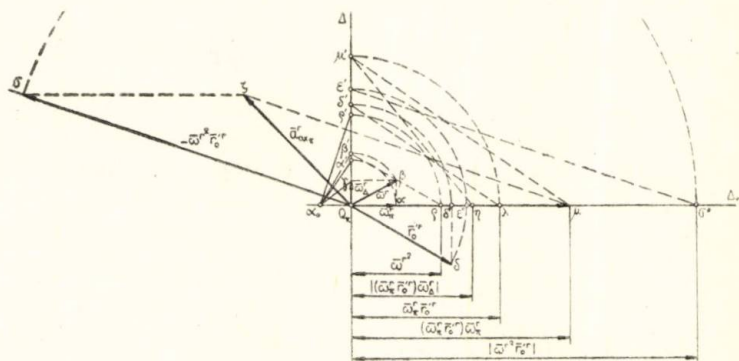


Fig. 7

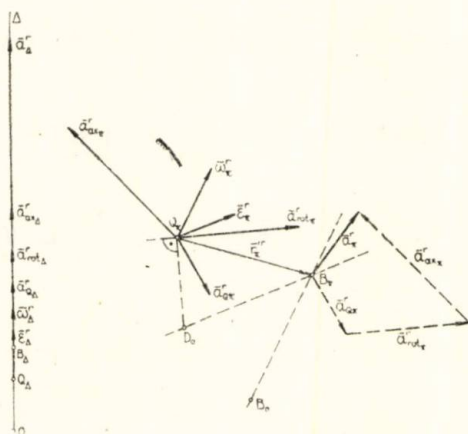


Fig. 8

Dans la fig. 8 sont déterminées les composantes plane-axielles, $\bar{a}_{\Pi}^r$ et $\bar{a}_{\Delta}^r$, de l'accélération du point B, qui permettront ensuite d'obtenir finalement, l'accélération

$$(2.7) \quad \bar{a}_B = \bar{a} = \sigma_a(\bar{a}_{\Pi}^r + \bar{a}_{\Delta}^r).$$

3. Détermination graphique du centre instantané des accélérations

La condition d'existence du centre instantané des accélérations, I , est :

$$(3.1) \quad \bar{a}_I = \bar{a}_Q + \bar{\varepsilon} \times \bar{r}'_I + \bar{\omega} \times (\bar{\omega} \times \bar{r}'_I) = 0,$$

d'où on tirera le terme $\bar{\varepsilon} \times \bar{r}'_I$. Après la multiplication vectorielle de ce terme avec $\bar{\varepsilon}$, on obtiendra la relation

$$(3.2) \quad \bar{\varepsilon} \times \bar{a}_Q + \bar{\omega}^2 \bar{a}_Q - (\varepsilon^2 + \omega^4) \bar{r}'_I + (\bar{\varepsilon} \bar{r}'_I) \bar{\varepsilon} + (\bar{\omega} \bar{r}'_I) [\bar{\varepsilon} \times \bar{\omega} + \bar{\omega}^2 \bar{\omega}] = 0.$$

La condition vectorielle (3.1) est équivalente à un système de trois équations scalaires de projection sur trois directions ne se trouvant pas dans le même plan, qui peuvent être, par exemple, les direction des vecteurs $\bar{\omega}$, $\bar{\varepsilon}$ et $\bar{\omega} \times \bar{\varepsilon}$.

On obtient ainsi les égalités

$$(3.3) \quad \begin{cases} \bar{a}_I \bar{\omega} = \bar{\omega} \bar{a}_Q + \bar{\omega} (\bar{\varepsilon} \times \bar{r}'_I) = 0, \\ \bar{a}_I \bar{\varepsilon} = \bar{\varepsilon} \bar{a}_Q + (\bar{\omega} \bar{r}'_I) (\bar{\omega} \bar{\varepsilon}) - \bar{\omega}^2 (\bar{\varepsilon} \bar{r}'_I) = 0, \\ \bar{a}_I (\bar{\omega} \times \bar{\varepsilon}) = \bar{a}_Q (\bar{\omega} \times \bar{\varepsilon}) + (\bar{\omega} \times \bar{\varepsilon}) (\bar{\varepsilon} \times \bar{r}'_I) - \bar{\omega}^2 \bar{r}'_I (\bar{\omega} \times \bar{\varepsilon}) = 0, \end{cases}$$

qui permettent de trouver les valeurs des produits scalaires qui figurent dans la relation (3.2), c'est-à-dire

$$(3.4) \quad \begin{cases} (\bar{\omega} \bar{r}'_I) = \frac{(\bar{\omega} \bar{\varepsilon}) (\bar{\varepsilon} \bar{a}_Q) + \bar{\omega}^2 [\bar{a}_Q (\bar{\omega} \times \bar{\varepsilon})] + \bar{\omega}^4 (\bar{\omega} \bar{a}_Q)}{(\bar{\omega} \times \bar{\varepsilon})^2} = b, \\ (\bar{\varepsilon} \bar{r}'_I) = \frac{\varepsilon^2 (\bar{\varepsilon} \bar{a}_Q) + (\bar{\omega} \bar{\varepsilon}) [\bar{a}_Q (\bar{\omega} \times \bar{\varepsilon})] + \bar{\omega}^2 (\bar{\omega} \bar{\varepsilon}) (\bar{\omega} \bar{a}_Q)}{(\bar{\omega} \times \bar{\varepsilon})^2} = c, \end{cases}$$

b et c pouvant être déterminés à l'aide des constructions graphiques présentées dans [3].

Tout en introduisant (3.4) en (3.2) on obtiendra l'expression du vecteur $\bar{r}'_I$ du centre instantané des accélérations

$$(3.5) \quad \bar{r}'_I = \frac{\bar{\varepsilon} \times \bar{a}_Q + \bar{\omega}^2 \bar{a}_Q}{\varepsilon^2 + \omega^4} + \frac{c}{\varepsilon^2 + \omega^4} \bar{\varepsilon} + \frac{b}{\varepsilon^2 + \omega^4} [\bar{\varepsilon} \times \bar{\omega} + \bar{\omega}^2 \bar{\omega}].$$

Si l'on emploie les notations $\bar{k}^r$ et $\bar{l}^r$ pour les vecteurs représentatifs des termes

$$(3.6) \quad \bar{k} = \bar{\varepsilon} \times \bar{a}_0 + \omega^2 \bar{a}_0, \quad \bar{l} = \bar{\varepsilon} \times \bar{\omega} + \bar{\omega}^2 \bar{\omega}$$

et c^r et b^r pour les segments représentatifs des grandeurs scalaires c et b , on obtiendra les composantes plane-axielles du vecteur représentatif du vecteur de position $\bar{r}_I^r$, déterminées par les relations

$$(3.7) \quad \begin{cases} \bar{r}_I^r = \frac{1}{\varepsilon^2 + \omega^4} [\bar{k}_I^r + c^r \bar{\varepsilon}_I^r + b^r \bar{l}_I^r], \\ \bar{r}_{I\Delta}^r = \frac{1}{\varepsilon^2 + \omega^4} [\bar{k}_{I\Delta}^r + c^r \bar{\varepsilon}_{I\Delta}^r + b^r \bar{l}_{I\Delta}^r]. \end{cases}$$

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DISTRIBUȚII DE ACCELERAȚII DETERMINATE PRIN METODA COORDONATELOR VECTORIALE PLAN-AXIALE

(Rezumat)

Introducind noțiunile de *componente plan-axiale ale parametrilor cinematici de ordinul doi* ai mișcării generale a solidului, definite prin relațiile (1.1)–(1.3), se aplică metoda coordonatelor vectoriale plan-axiale, elaborată de primii dintre autori, la determinarea distribuțiilor de accelerații în mișcarea generală a solidului.

Metoda este apoi aplicată la determinarea poziției centrului instantaneu al accelerațiilor, pentru al cărui vector de poziție în raport cu polul invariabil legat de solid se găsește expresia (3.5).

НЕКОТОРЫЕ ВОПРОСЫ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ (II)

В. М. СТАРЖИНСКИЙ

0. Введение

В первой части работы [1] рассматриваются колебания нелинейных автономных систем с произвольным числом $(k + 1)$ степеней свободы и близких к консервативным. Проводится преобразование возмущенной системы, при котором невозмущенная система может быть преобразована в квазилинейную неавтономную систему $2k$ -го порядка.

Если порождающее решение известно, то первая поправка соответствующего решения возмущенной системы может быть определена из системы уравнений в вариациях Пуанкаре. Если порождающее решение общее, то интегрирование последней сводится к квадратурам, что и показано на примере механической системы с одной степенью свободы.

1. Уравнение Дюффинга с линейным демпфированием

Названное уравнение имеет вид

$$(1.1) \quad \ddot{u} + 2\varepsilon\dot{u} + u + \alpha u^3 = 0, \quad (\alpha > 0, \varepsilon > 0),$$

т. е. (см. [1, (3.1) и (3.2)]) $f(\dot{u}) = \dot{u}$, $U(u) = -\alpha u^3$, $V(u) = \frac{1}{4} \alpha u^4$.

Условие [1, (1.7)] примет вид

$$|1 + \alpha \rho^2 \sin^4 \theta + \varepsilon \sin 2\theta| > 0$$

и заведомо выполнено при $0 \leq \varepsilon \leq 1$. Интеграл энергии [1, (3.6)] невозмущенного ($\varepsilon = 0$) уравнения (1.1) примет вид

$$\rho^2 + \frac{1}{2} \alpha \rho^4 \sin^4 \theta = \mu^2$$

и при условии $2\alpha\mu^2 \sin^4 \theta < 1$, которое заведомо выполнено при

$$(1.2) \quad \mu^2 < \frac{1}{2\alpha}$$

допускает единственное решение [1, (3.7)]

$$(1.3) \quad \rho = \rho_0(\theta; \mu) = \left[1 - \frac{1}{4} \alpha \mu^2 \sin^4 \theta + O(\mu^4) \right].$$

Вычислим [1, (3.8) и (3.9)]

$$\frac{\partial \rho_0}{\partial \mu} = 1 - \frac{3}{4} \alpha \mu^2 \sin^4 \theta + O(\mu^4),$$

$$\begin{aligned} \left(\frac{\partial f}{\partial \varepsilon} \right)_0 &= -2\rho_0 \cos^2 \theta (1 + \alpha \rho_0^2 \sin^4 \theta)^{-1} [1 - \alpha \rho_0^2 \sin^4 \theta (1 + \\ &+ \alpha \rho_0^2 \sin^4 \theta)^{-1}] = -2 \cos^2 \theta \left[1 - \frac{9}{4} \alpha \mu^2 \sin^4 \theta + O(\mu^4) \right]. \end{aligned}$$

Будем искать решение, удовлетворяющее начальным условиям: $u(0) = 0$, $\dot{u}(0) = \dot{u}_0 > 0$, при этом в силу [1, (3.3)] и (4.3) $\theta_0 = 0$ и $\mu = \dot{u}_0$. Предположим также (см. (1.2)), что $\dot{u}_0 < 1/\sqrt{2\alpha}$.

По формуле [1, (2.7)] найдем

$$\begin{aligned} \rho_1(\theta; \mu) &= -\mu \left[\theta + \frac{1}{2} \sin 2\theta + \frac{1}{4} \alpha \mu^2 \left(-\frac{3}{4} \theta + \frac{3}{16} \sin 4\theta + \right. \right. \\ &\left. \left. + \frac{1}{4} \sin^3 2\theta - \frac{3}{2} \sin 2\theta \sin^4 \theta - 3\theta \sin^4 \theta \right) + O(\mu^4) \right]. \end{aligned}$$

Из второго уравнения [1, (3.4)] найдем

$$\begin{aligned} t &= \int_0^\theta (1 - \alpha \mu^2 \sin^4 \theta - \varepsilon \sin 2\theta + \dots) d\theta = \theta - \varepsilon \sin^2 \theta - \\ &- \alpha \mu^2 \left(\frac{1}{32} \sin 4\theta - \frac{1}{4} \sin 2\theta + \frac{3}{8} \theta \right) + \dots, \end{aligned}$$

где точками обозначены члены порядка ε^2 и $\varepsilon \mu^2$. Обращая последнюю формулу, получим

$$\theta = t + \varepsilon \sin^2 t + \frac{1}{4} \alpha \mu^2 \left(\frac{3}{2} t - \sin 2t + \frac{1}{8} \sin 4t \right) + \dots$$

Выпишем решение в виде [1, (3.10)]

$$(1.4) \quad u = \left[\mu \left(1 - \frac{1}{4} \alpha \mu^2 \sin^4 t \right) - \varepsilon \mu \left(t + \frac{1}{2} \sin 2t \right) + O(\varepsilon^2) \right] \times \\ \times \sin \left[t + \varepsilon \sin^2 t + \frac{1}{4} \alpha \mu^2 \left(\frac{3}{2} t - \sin 2t + \frac{1}{8} \sin 4t \right) + O(\varepsilon^2, \varepsilon \mu^2) \right]$$

для интервала изменения t порядка $O(1/\varepsilon)$.

Замечание. Используя специфику примера, именно линейность демпфирования, можно получить решение справедливое для всех $t > 0$. Подстановка

$$\mu = e^{-\varepsilon t} v$$

приведет уравнение (1.1) к виду

$$(1.5) \quad \frac{dx}{dt} = g(t, v; \alpha), \quad x = \begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \quad g = \begin{pmatrix} \dot{v} \\ -(1 - \varepsilon^2)v - \alpha e^{-2\varepsilon t} v^3 \end{pmatrix}.$$

Будем искать его общее решение в виде

$$x = x_0(t) + \alpha x_1(t) + \dots$$

Общее решение невозмущенного (т. е. при $\alpha = 0$) уравнения (4.5) при $0 < \varepsilon < 1$ (аналогично рассматриваются случаи $\varepsilon = 1$ и $\varepsilon > 1$) есть

$$x_0 = \begin{pmatrix} M \cos Rt + N \sin Rt \\ -RM \sin Rt + RN \cos Rt \end{pmatrix}, \quad (R \equiv \sqrt{1 - \varepsilon^2}).$$

Формула [1, (2.7)] дает нам для первой поправки $x_1(t)$

$$(1.6) \quad x_1 = \left(\frac{\partial x_0}{\partial a} \right)^{-1} \int_0^t \left(\frac{\partial x_0}{\partial a} \right)^{-1} \left(\frac{\partial g}{\partial \alpha} \right)_0 dt,$$

где a вектор с компонентами M и N , а

$$\frac{\partial x_0}{\partial a} = \begin{vmatrix} \frac{\partial v}{\partial M} & \frac{\partial v}{\partial N} \\ \frac{\partial \dot{v}}{\partial M} & \frac{\partial \dot{v}}{\partial N} \end{vmatrix}$$

и индекс нуль означает подстановку $\alpha = 0$, $x = x_0(t)$. Опуская очевидные вычисления в формуле (1.6), выпишем общее решение уравнения (1.1) при $0 < \varepsilon < 1$

$$(1.7) \quad u = [M \cos Rt + N \sin Rt + \alpha v_1(t) + O(\alpha^2)] e^{-\varepsilon t},$$

где

$$\begin{aligned} v_1(t) = & \frac{1}{16R} \{ 2[e^{-2\varepsilon t} (-\varepsilon \sin 2Rt - R \cos 2Rt) + R] \times \\ & \times [M(M^2 + 3N^2) \cos Rt - N(N^2 + 3M^2) \sin Rt] + \\ & + (4 - 3\varepsilon^2)^{-1} [e^{-2\varepsilon t} (-\varepsilon \sin 4Rt - 2R \cos 4Rt) + 2R] [M(M^2 - \\ & - 3N^2) \cos Rt + N(N^2 - 3M^2) \sin Rt] + (4 - 3\varepsilon^2)^{-1} [e^{-2\varepsilon t} (-\varepsilon \cos 4Rt + \\ & + 2R \sin 4Rt) + \varepsilon] [N(N^2 - 3M^2) \cos Rt - M(M^2 - 3N^2) \sin Rt] - \\ & - 4[e^{-2\varepsilon t} (-\varepsilon \cos 2Rt + R \sin 2Rt) + \varepsilon] [N^3 \cos Rt + M^3 \sin Rt] + \\ & + 3(M^2 + N^2) (1 - e^{-2\varepsilon t}) (N \cos Rt - M \sin Rt). \end{aligned}$$

Очевидно $M = u(0)$, $N = (1 - \varepsilon^2)^{-1/2} \dot{u}(0)$.

2. Пружинный маятник с демпфированием

Рассмотрим плоский пружинный маятник массы m на невесомой пружине длиной l в ненапряженном состоянии и подчиняющейся закону Гука с жесткостью c (рис. 1). Пусть x и $y' = l + \lambda + y$ — декар-

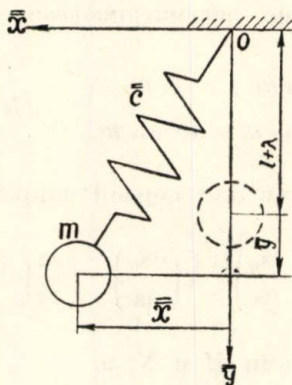


Рис. 1

товы координаты массы m , отсчитываемые от точки подвеса O , где $\lambda = mg/c$ — статическое удлинение пружины. Выберем постоянную потенциальную энергию Π силы тяжести и упругой силы пружины таким

образом, чтобы она обращалась в нуль для положения статического равновесия ($x = y = 0$), тогда

$$\Pi = -mgy + \frac{1}{2} c [\sqrt{x^2 + (l + \lambda + y)^2} - l]^2 - \frac{1}{2} c \lambda^2,$$

$$T = \frac{1}{2} m \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right].$$

Допустим на массу m действует еще сила сопротивления R , пропорциональная скорости: $R = -bv$. Обозначим через $\omega = \sqrt{c/m}$ круговую частоту вертикальных колебаний массы на пружине и введем безразмерные время $\tau = \omega t$ и координаты $\xi = x/l$, $\eta = y/l$. Тогда уравнения движения запишутся в виде

$$(2.1) \quad \begin{cases} \frac{d^2 \eta}{d\tau^2} + \eta - \{(1 + \gamma + \eta) [\xi^2 + (1 + \gamma + \eta)^2]^{-1/2} - 1\} = -2\varepsilon \frac{d\eta}{d\tau}, \\ \frac{d^2 \xi}{d\tau^2} + \frac{\gamma}{1 + \gamma} \xi - \left\{ \xi [\xi^2 + (1 + \gamma + \eta)^2]^{-1/2} - \frac{\xi}{1 + \gamma} \right\} = -2\varepsilon \frac{d\xi}{d\tau}, \end{cases}$$

где

$$\gamma = \frac{\lambda}{l}, \quad \varepsilon = \frac{b}{2\sqrt{cm}}$$

безразмерные параметры, а разложения фигурных скобок в окрестности $\xi = \eta = 0$ начинаются с членов второй степени. При $\varepsilon = 0$ имеет место интеграл энергии

$$(2.2) \quad \frac{1}{cl} \sqrt{2(T + \Pi)c} = \mu = \text{const.}$$

Подстановка Ляпунова

$$(2.3) \quad \eta = \rho \sin \theta, \quad \frac{d\eta}{d\tau} = \rho \cos \theta, \quad \xi = \rho z_1, \quad \frac{d\xi}{d\tau} = \rho z_2$$

приведет возмущенную систему (2.4) и интеграл (2.2) невозмущенной системы к виду [1, (4.8) и (4.6)]:

$$(2.4) \quad \begin{cases} \frac{d\rho}{d\theta} = \frac{U \cos \theta - 2\varepsilon \rho \cos^2 \theta}{1 - \rho^{-1} U \sin \theta + \varepsilon \sin 2\theta}, \\ \frac{dz_1}{d\theta} = \frac{z_2 - z_1 \rho^{-1} U \cos \theta + 2\varepsilon z_1 \cos^2 \theta}{1 - \rho^{-1} U \sin \theta + \varepsilon \sin 2\theta}, \\ \frac{dz_2}{d\theta} = \frac{-\gamma(1 + \gamma)^{-1} z_1 - z_2 \rho^{-1} U \cos \theta + \rho^{-1} V - 2\varepsilon z_2 \sin^2 \theta}{1 - \rho^{-1} U \sin \theta + \varepsilon \sin 2\theta}, \end{cases}$$

$$(2.5) \quad \rho^2 \left[1 + \frac{\gamma}{1+\gamma} z_1^2 + z_2^2 + \rho S(\rho, \theta, z_1, z_2) \right] = \mu^2,$$

где

$$U = -\frac{\rho^2 z_1^2}{2(1+\gamma)^2} + O(\rho^3), \quad V = -\frac{\rho^2 z_1 \sin \theta}{(1+\gamma)^2} + O(\rho^3)$$

и предполагается выполненным условие [1, (1.7)], т. е.

$$(2.6) \quad \left| \frac{1}{2} (1+\gamma)^{-2} \rho z_1^2 \sin \theta + \varepsilon \sin 2\theta \right| < 1.$$

Невозмущенная система (2.4) (т. е. при $\varepsilon = 0$) допускает [2] порождающее решение

$$(2.7) \quad \begin{cases} \rho_0(\theta; \mu, M, N) = \mu[1 + g^2(M^2 + N^2)]^{-1/2} + O(\mu^2), \\ z_1^0(\theta; \mu, M, N) = M \cos g\theta + N \sin g\theta + O(\mu), \\ z_2^0(\theta; \mu, M, N) = g(-M \sin g\theta + N \cos g\theta) + O(\mu), \end{cases}$$

где

$$g = \sqrt{\frac{\gamma}{1+\gamma}} \quad \left(g \neq \frac{1}{2} \right).$$

Это решение, как показано в [2], является общим при всех γ , кроме $\gamma = \frac{1}{3}$ ($g = \frac{1}{2}$). В нашем примере векторы $\mathbf{a}$ и $\mathbf{x}_0$ суть

$$\mathbf{a} = \begin{pmatrix} \mu \\ M \\ N \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} \rho_0 \\ z_1^0 \\ z_2^0 \end{pmatrix}.$$

Вычислим элементы формулы [1, (2.7)]:

$$\frac{\partial x_0}{\partial a} = \left\| \begin{array}{ccc} K^{-1/2} + O(\mu) & -g^2 M K^{-3/2} \mu + O(\mu^2) & -g^2 N K^{-3/2} \mu + O(\mu^2) \\ \varphi(\theta) + O(\mu) & \cos g\theta + O(\mu) & \sin g\theta + O(\mu) \\ \psi(\theta) + O(\mu) & -g \sin g\theta + O(\mu) & g \cos g\theta + O(\mu) \end{array} \right\|,$$

где

$$K = 1 + g^2(M^2 + N^2).$$

При вычислении обратной матрицы несколько огрубим результат, положив $\varphi(\theta) = \psi(\theta) = 0$, тогда получим для $(\partial \mathbf{x}_0 / \partial \mathbf{a})^{-1}$ выражение

$$\left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} \right)^{-1} =$$

$$= \begin{vmatrix} K^{1/2} + O(\mu) & g^2 K^{-1} \mu (M \cos g\theta + N \sin g\theta) + O(\mu^2) & g K^{-1} \mu (-M \sin g\theta + N \cos g\theta) + O(\mu^2) \\ O(\mu) & \cos g\theta + O(\mu) & -g^{-1} \sin g\theta + O(\mu) \\ O(\mu) & \sin g\theta + O(\mu) & g^{-1} \cos g\theta + O(\mu) \end{vmatrix}.$$

Вектор $(\partial \mathbf{f} / \partial \varepsilon)_0$, где $\mathbf{f}$ есть вектор правых частей системы (2.4), а индекс нуль означает подстановку $\varepsilon = 0$ и порождающего решения (2.7), равен

$$\left(\frac{\partial \mathbf{f}_1}{\partial \varepsilon} \right)_0 = \begin{pmatrix} -2K^{-1/2} \mu \cos^2 \theta + O(\mu^2) \\ 2(M \cos g\theta + N \sin g\theta) \cos^2 \theta - g(-M \sin g\theta + N \cos g\theta) \sin 2\theta + O(\mu) \\ -2g(N \cos g\theta - M \sin g\theta) \sin^2 \theta + g^2(M \cos g\theta + N \sin g\theta) \sin 2\theta + O(\mu) \end{pmatrix}.$$

Мы положим в формуле [1, (2.7)] $\theta_0 = 0$, согласно (2.3) это означает, что в начальный момент $\tau = 0$ имеем $\eta(0) = 0$. Опуская очевидные вычисления по формуле [1, (2.7)], выпишем нужные нам первые две компоненты вектора $\mathbf{x}_1(\theta)$ первой поправки:

$$\rho_1 = -K^{-1/2} \mu \left\{ \theta + \frac{1}{2} \sin 2\theta + g^2 K^{-1} \left[-gMN \sin^2 \theta - \right. \right.$$

$$\left. - \frac{1}{2} g(1-g)^{-1} MN \sin^2(1-g)\theta - \frac{1}{2} g(1+g)^{-1} MN \sin^2(1+g)\theta - \right.$$

$$\left. - \frac{1}{4} g(1-g)^{-1} N^2 \sin 2(1-g)\theta + \frac{1}{4} g^2(1+g) \sin 2(1+g)\theta \right] \Big\} + O(\mu^2),$$

$$z_1^1 = -2M\theta \cos g\theta - \frac{1}{2} M \sin 2\theta \cos g\theta + \frac{1}{2} gN \cos 2\theta \cos g\theta -$$

$$- \frac{1}{2} N \sin 2\theta \cos g\theta - \frac{1}{2} (1-g^2) g^{-1} \cos g\theta +$$

$$+ \frac{1}{4} g(1-g)^{-1} M \cos 2(1-g)\theta \sin g\theta + \frac{1}{4} (1+g)^{-1} \cos 2(1+g)\theta \sin g\theta -$$

$$- \frac{1}{4} g(1-g)^{-1} M \sin g\theta - \frac{1}{4} g(1+g)^{-1} M \sin g\theta - 2N\theta \sin g\theta +$$

$$+ \frac{1}{4} g(1-g)^{-1} N \sin 2(1-g)\theta \sin g\theta +$$

$$+ \frac{1}{4} g(1+g)^{-1} N \sin 2(1+g)\theta \sin g\theta + O(\mu).$$

Остается проинтегрировать второе уравнение [1, (1.5)], это нам даст опять-таки при $\theta_0 = 0$

$$\theta = \theta(\tau) = \tau + \varepsilon \sin^2 \tau + \dots$$

где точками отмечены члены второго порядка относительно ε и μ . Окончательно получим решение системы (5.1) для случая $\eta(0) = 0$ в виде

$$\eta(\tau) = [\rho_0(\theta(\tau); \mu, M, N) + \varepsilon \rho_1(\theta(\tau); \mu, M, N)] \sin(\tau + \varepsilon \sin^2 \tau) + O(\varepsilon^2),$$

$$\begin{aligned} \xi(\tau) = & [\rho_0(\theta(\tau); \mu, M, N) + \varepsilon \rho_1(\theta(\tau); \mu, M, N)] \times \\ & \times [\zeta_1^0(\theta(\tau); \mu, M, N) + \varepsilon \zeta_1^1(\theta(\tau); \mu, M, N)] + O(\varepsilon^2). \end{aligned}$$

Постоянные μ (начальное значение энергии [1, (3.2)]), M и N определяются из начальных условий $\eta'(0)$, $\xi(0)$ и $\xi'(0)$; интервал изменения τ имеет порядок $O(1/\varepsilon)$.

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UNELE PROBLEME ALE TEORIEI OSCILAȚIILOR NELINIARE (II)

(Rezumat)

În prima parte a lucrării [1] s-a expus o metodă de transformare care permite trecerea unui sistem neliniar autonom cu $2(k+1)$ grade de libertate, într-un sistem neautonom cuasiliniar de ordinul $2k$ și s-a arătat că atunci când se cunoaște soluția generală a sistemului neperturbat, integrarea sistemului de ecuații în variații se reduce la cuadraturi.

În lucrarea de față se ilustrează procedeul pe cazul ecuației Duffing cu amortizare liniară și pe exemplul oferit de pendulul elastic din fig. 1.

A NOTE ON SIMILARITY SOLUTIONS BY GROUP TRANSFORMS

BY

KRISHNA LAL

1. Introduction

The similarity equations for the power law fluid flow in the wake behind a flat plate have been obtained by the applications of linear and spiral group transforms. It has been shown that the linear group transform equation may be reduced to the equation and solution for Newtonian fluid flow which is known earlier [1].

2. Basic Equations

Consider a flat plate, placed in the direction of x -axis and y -axis perpendicular to the plate. The equation of motion in the wake behind a flat plate for Newtonian fluid is obtained [1] by putting

$$(1) \quad w(x, y) = U - u(x, y)$$

into

$$(2) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2},$$

where the symbols have their usual meanings and quadratic terms in w are neglected. Thus for the power law fluids flow where the stress and strain relations are connected by

$$(3) \quad \sigma_{ij} = k \left| \sum_{m=1}^3 \sum_{l=1}^3 e_{lm} e_{ml} \right|^{\frac{n-1}{2}} e_{ij},$$

$$(4) \quad e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},$$

we have the equation of motion giving velocity distribution in the wake behind the plate by putting equation (1) into

$$(5) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial}{\partial y} \left\{ \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right\}$$

and neglecting the quadratic terms.

Thus finally, we have

$$(6) \quad U \frac{\partial w}{\partial x} = (-1)^{n-1} \nu \frac{\partial}{\partial y} \left\{ \left| \frac{\partial w}{\partial y} \right|^n \right\}.$$

To determine the similarity solutions of equation (6) with the boundary conditions

$$(7) \quad \begin{cases} y = 0 & : & \frac{\partial w}{\partial y} = 0, \\ y = \infty & : & w = 0, \end{cases}$$

we put

$$k_1 = \frac{(-1)^{n-1} \nu}{U}$$

for simplicity of calculations.

In this note following two group transformations are used to determine the similarity solutions.

3. Linear Transformation

In this case we put

$$(8) \quad w = A^{\alpha_1} \bar{w}, \quad x = A^{\alpha_2} \bar{x}, \quad y = A^{\alpha_3} \bar{y}$$

into

$$(9) \quad \frac{\partial w}{\partial x} = k_1 \frac{\partial}{\partial y} \left\{ \left| \frac{\partial w}{\partial y} \right|^n \right\},$$

where A , α_1 , α_2 , α_3 are certain constants.

Thus for invariance purposes, we have

$$(10) \quad \alpha_1 - \alpha_2 = n(\alpha_1 - \alpha_3) - \alpha_3$$

or

$$b = \frac{m(n+1)-1}{n-1},$$

where

$$b = \frac{\alpha_1}{\alpha_2}, \quad m = \frac{\alpha_3}{\alpha_2}.$$

Substituting equations of set (8) with the values of m and b , we have the following absolute invariants defined by

$$(11) \quad \eta = \frac{y}{x^m}, \quad w = x^b g(\eta).$$

Substituting the equations of set (11) into (9), we have

$$(12) \quad k_1 n g'' g'^{(n-1)} + m \eta g' - b g = 0.$$

For $n = 1$, we have from equation (10), that

$$\alpha_2 = 2\alpha_3 \quad \text{and thus} \quad \frac{y}{\sqrt{x}} = \frac{\bar{y}}{\sqrt{\bar{x}}}.$$

Hence for the invariance purposes, we get

$$(13) \quad \begin{cases} \eta = \frac{y}{\sqrt{x}}, \\ w = x^r g(\eta), \end{cases}$$

where r is a constant.

Substituting the Eqs. (13) into (9) for $n = 1$, we have

$$(14) \quad \frac{v}{U} g'' + \frac{1}{2} \eta g' - r g = 0,$$

whose solution for $r = -\frac{1}{2}$ is

$$(15) \quad g = e^{-\eta^2/4}$$

and has been discussed earlier [1].

4. Spiral Transformations

In this case we substitute

$$(16) \quad x = \bar{x} + \beta_1 b, \quad y = e^{\beta_2 b} \bar{y}, \quad w = e^{\beta_3 b} \bar{w}$$

into Eq. (9), where $\beta_1, \beta_2, \beta_3$ and b are certain constants.

Simplifying for invariance purposes, we have

$$(17) \quad \beta_2 = \frac{n-1}{n+1} \beta_3 \quad \text{or} \quad p = \frac{\beta_2}{\beta_1} = q \frac{n-1}{n+1},$$

where $q = \frac{\beta_3}{\beta_1}$.

Thus the absolute invariants are easily determined as

$$(18) \quad \eta = \frac{y}{e^{xp}}, \quad w = e^{\frac{p(n+1)x}{n-1}} F'(\eta).$$

Substituting the equations of set (18) into (9), we have

$$(19) \quad nk_1 F'' F'^{(n-1)} + p\eta F' - qF = 0,$$

where F' and F'' are the derivatives with respect to η .

Thus the similar solutions exist for two possible groups of transformations which were used by Na [2] in another problem of similarity solutions. Hayasi [3] has also discussed the possible similar solutions for two dimensional and axisymmetric flow. As far known to the author, the present problem has not been discussed previously.

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NOTĂ ASUPRA SIMILARITĂȚII SOLUȚIILOR PRIN TRANSFORMĂRI GRUPALE

(Rezumat)

Se obțin ecuațiile de similaritate pentru curgerea unui fluid nenewtonian, de tip exponențial, îndărătul unei plăci plane. Se arată că prin transformarea liniară (8), ecuația de mișcare se reduce la forma cunoscută (14), care descrie curgerea similară a unui fluid newtonian.

HYDROMAGNETIC FLOW IN A ROTATING PIPE (II)

BY

M. S. SASTRY

1. Introduction

The general problem of Hydromagnetic flow in a straight rotating pipe of finite thickness and arbitrary cross-section was formulated by the author [1]. In this paper we study the particular problem of flow in a circular pipe. It is found that the effect of the magnetic field and the rotation is to reduce the mass flux through the pipe. The somewhat limited case of a parallel sided duct of perfectly conducting walls has been considered by N a d n a and M o h a n t y [2].

The basic equations deduced by the author [1] are

$$(1) \quad \nabla^4 \Psi + \mathcal{T} \frac{\partial W}{\partial X} + \mathcal{M}^2 J \begin{pmatrix} \Phi & \nabla^2 \Phi \\ X & Y \end{pmatrix} - \mathcal{R} J \begin{pmatrix} \Psi & \nabla^2 \Psi \\ X & Y \end{pmatrix} = 0,$$

$$(2) \quad \nabla^2 W + 4 + \mathcal{M}^2 J \begin{pmatrix} \Phi & B_z \\ X & Y \end{pmatrix} - \mathcal{T} \frac{\partial \Psi}{\partial X} - \mathcal{R} J \begin{pmatrix} \Psi & W \\ X & Y \end{pmatrix} = 0,$$

$$(3) \quad \nabla^2 \Phi + R_m J \begin{pmatrix} \Phi & \Psi \\ X & Y \end{pmatrix} = 0,$$

$$(4) \quad \nabla^2 B_z + \mathcal{R}_m \left[J \begin{pmatrix} \Phi & W \\ X & Y \end{pmatrix} + J \begin{pmatrix} B_z & W \\ X & Y \end{pmatrix} \right] = 0.$$

2. Solution of the Equations for a Circular Pipe

Since these equations are non-linear partial and simultaneous it is not possible to obtain a closed form solution. However by assuming $\mathcal{R}_m$ and $\mathcal{T}$ to be small and of the same order of smallness we can obtain useful solutions by using a regular perturbation scheme.

We write

$$(5) \quad \begin{cases} \Phi = \Phi_0 + \mathcal{T}\Phi_1 + \mathcal{T}^2\Phi_2 + \dots, \\ \Psi = \Psi_0 + \mathcal{T}\Psi_1 + \mathcal{T}^2\Psi_2 + \dots, \\ W = W_0 + \mathcal{T}W_1 + \mathcal{T}^2W_2 + \dots, \\ B_z = B_{z_0} + \mathcal{T}B_{z_1} + \mathcal{T}^2B_{z_2} + \dots \end{cases}$$

We will consider the flow in a circular pipe with internal radius a and external radius λa or in dimensionless form 1 and λ respectively. It is convenient to change to polar coordinates with

$$X = r \cos \theta, \quad Y = r \sin \theta,$$

$$\frac{\partial}{\partial X} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial Y} = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta},$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},$$

$$\begin{aligned} \nabla^4 = & \frac{\partial^4}{\partial r^4} + \frac{2}{r} \frac{\partial^3}{\partial r^3} + \frac{1}{r^3} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + \frac{2}{r^2} \frac{\partial^4}{\partial r^2 \partial \theta^2} - \\ & - \frac{2}{r^3} \frac{\partial^3}{\partial \theta^2 \partial r} + \frac{4}{r^4} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^4} \frac{\partial^4}{\partial \theta^4}. \end{aligned}$$

Substituting Eqs. (5) in Eqs. (1) to (4) and equating the coefficients of various powers of $\mathcal{T}$ we obtain the following sets of equations:

Zero order approximation:

$$(6a) \quad \nabla^2 \Phi_0 = 0, \quad (6b) \quad \nabla^4 \Psi_0 = 0,$$

$$(6c) \quad \nabla^2 B_{z_0} = 0, \quad (6d) \quad \nabla^2 W_0 + 4 = 0.$$

First order approximation:

$$(7a) \quad \nabla^2 \Phi_1 = 0, \quad (7b) \quad \nabla^4 \Psi_1 + \frac{\partial W_0}{\partial X} = 0,$$

$$(7c) \quad \nabla^2 B_{z_1} + \mathcal{E} \left[\frac{\partial \Phi_0}{\partial X} \frac{\partial W_0}{\partial Y} - \frac{\partial \Phi_0}{\partial Y} \frac{\partial W_0}{\partial X} \right] = 0,$$

$$(7d) \quad \nabla^2 W_1 + \mathcal{M}^2 \left[\frac{\partial \Phi_0}{\partial X} \frac{\partial B_{z_1}}{\partial Y} - \frac{\partial \Phi_0}{\partial Y} \frac{\partial B_{z_1}}{\partial X} \right] - \mathcal{Q} \left[\frac{\partial \Psi_1}{\partial X} \frac{\partial W_0}{\partial Y} - \frac{\partial \Psi_1}{\partial Y} \frac{\partial W_0}{\partial X} \right] = 0.$$

Second order approximation:

$$(8a) \quad \nabla^2 \Phi_2 + \in \left[\frac{\partial \Phi_0}{\partial X} \frac{\partial \Psi_1}{\partial Y} - \frac{\partial \Psi_1}{\partial X} \frac{\partial \Phi_0}{\partial Y} \right] = 0,$$

$$(8b) \quad \nabla^4 \Psi_2 + \frac{\partial W_1}{\partial X} + \mathcal{M}^2 \left[\frac{\partial \Phi_0}{\partial X} \frac{\partial}{\partial Y} (\nabla^2 \Phi_2) - \frac{\partial \Phi_0}{\partial Y} \frac{\partial}{\partial X} (\nabla^2 \Phi_2) \right] - \\ - \mathcal{R} \left[\frac{\partial \Psi_1}{\partial X} \frac{\partial}{\partial Y} (\nabla^2 \Psi_1) - \frac{\partial \Psi_1}{\partial Y} \frac{\partial}{\partial X} (\nabla^2 \Psi_1) \right] = 0,$$

$$(8c) \quad \nabla^2 B_{z_2} + \in \left[\frac{\partial \Phi_0}{\partial X} \frac{\partial W_1}{\partial Y} - \frac{\partial \Phi_0}{\partial Y} \frac{\partial W_1}{\partial X} + \frac{\partial \Phi_1}{\partial X} \frac{\partial W_0}{\partial Y} - \frac{\partial \Phi_1}{\partial Y} \frac{\partial W_0}{\partial X} \right] = 0,$$

$$(8d) \quad \nabla^2 W_2 + \mathcal{M}^2 \left[\frac{\partial \Phi_0}{\partial X} \frac{\partial B_{z_2}}{\partial Y} - \frac{\partial \Phi_0}{\partial Y} \frac{\partial B_{z_2}}{\partial X} + \frac{\partial \Phi_1}{\partial X} \frac{\partial B_{z_1}}{\partial Y} - \frac{\partial \Phi_1}{\partial Y} \frac{\partial B_{z_1}}{\partial X} \right] - \\ - \mathcal{R} \left[\frac{\partial \Psi_2}{\partial X} \frac{\partial W_0}{\partial Y} - \frac{\partial \Psi_2}{\partial Y} \frac{\partial W_0}{\partial X} + \frac{\partial W_1}{\partial Y} \frac{\partial \Psi_1}{\partial X} - \frac{\partial W_1}{\partial X} \frac{\partial \Psi_1}{\partial Y} \right] - \frac{\partial \Psi_1}{\partial X} = 0,$$

where $\in = \mathcal{R}_m / \mathcal{T}$ is of order 1.

The general boundary conditions which are essential to the complete specification of the solution of the preceding equations are essentially those given by Eqs. (18), (22), (23) and (24) of [1] together with the condition that all the physical variables should remain bounded at $r = 0$.

To solve the equations the θ dependence is removed by assuming a particular form of solution satisfying the boundary conditions as in B a - r u a [3]. Thus, the partial differential equations are transformed into ordinary differential equations which can be solved by standard methods. Eqs. (6a) to (6d) are for flow in a non-rotating pipe i. e., neglecting the effects of rotation and induced field and their solution is easily found to be

$$(9a) \quad \Psi_0 = 0, \quad (9b) \quad B_{z_0} = 0,$$

$$(9c) \quad \Phi_0 = -r \sin \theta, \quad (9d) \quad W_0 = (1 - r^2).$$

The differential sets (7a) to (7d) and (8a) to (8d) are then solved and the solutions are given in the appendix.

The mass flux through the pipe is given by

$$Q = \frac{ca^4 \nu}{4} \int_0^1 \int_0^{2\pi} (W_0 + \mathcal{T} W_1 + \mathcal{T}^2 W_2) dr d\theta, \\ = \pi ca^4 \nu \left[\frac{1}{8} - \frac{\mathcal{R}_m M^2 7k + 1}{384 k + 1} - \frac{\mathcal{T}^2}{768} \left(\frac{1}{24} + \frac{\mathcal{R}^2}{56 \cdot 768} \right) \right],$$

where

$$k = \frac{\sigma_2 \lambda^2 - 1}{\sigma_1 \lambda^2 + 1}.$$

Thus up to the order of approximation considered the effect of both the magnetic field and the rotation is to decrease the mass flow through the pipe. Thus in the presence of a magnetic field, for a given mass flow we require a larger pressure gradient.

The central plane perpendicular to the axis of rotation is given by $\theta = \pm \pi/2$. It can be seen from Eqs. (1) of appendix that $v_\theta = \partial \Psi / \partial r = 0$ for $\theta = \pm \pi/2$. Thus as in the non-magnetic case, the fluid particles from one half cannot go the other half.

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Appendix

$$\begin{aligned} \Psi = & \mathcal{T} \left[\frac{r(1-r^2)^2}{96} \cos \theta \right] + \mathcal{T}^2 \left[\frac{A_1 \mathcal{M}^2}{384} r(1-r^2)^2 \cos \theta - \right. \\ (1) \quad & - \frac{\mathcal{M}^2}{9 \cdot 216} \in (2r - 7r^3 + 8r^5 - 3r^7) \cos \theta + \frac{\mathcal{M}^2 \in}{15 \cdot 360} r^3 (1-r^2)^2 \cos 3\theta + \\ & + \frac{\mathcal{R}}{96 \cdot 72 \cdot 640} (17r^2 - 36r^4 + 20r^6 - r^{10}) \sin 2\theta], \\ W = & (1-r^2) + \mathcal{T} \left[\frac{A_1 \mathcal{M}^2}{4} (1-r^2) + \frac{\mathcal{M}^2 \in}{32} (1-r^4) + \frac{\mathcal{M}^2 \in}{48} r^2 (1-r^2) \cos 2\theta + \right. \\ & + \frac{\mathcal{R}}{2 \cdot 304} (3r - 6r^3 + 4r^5 - r^7) \sin \theta] + \mathcal{T}^2 \left[\mathcal{M}^2 \left\{ \frac{(24L + \alpha_1)}{96} (r^2 - r^4) + \right. \right. \\ & + \frac{\alpha_2}{96 \cdot 27} (9r^2 - 8r^6) \left. \right\} \cos 2\theta + \frac{\mathcal{M}^2 \alpha_2}{2 \cdot 880} (r^4 - r^6) \cos 4\theta + \\ & + \frac{\mathcal{M}^2 \alpha_3}{160 \cdot 72} (10r^3 - 15r^5 + 6r^7 - r^9) \sin 3\theta + \frac{\mathcal{M}^2}{946 \cdot 240} \{ 5760 p_s (1-r^3) + \\ & + \alpha_3 (43r - 60r^5 + 20r^7 - 3r^9) \} \sin \theta + \frac{\mathcal{M}^2 \mathcal{R} \in}{96 \cdot 96 \cdot 120} (-35r - 20r^5 + \\ (2) \quad & + 20r^7 + 3r^9) \sin \theta + \frac{\mathcal{R}^2}{96 \cdot 96 \cdot 480 \cdot 420} (-923r^2 + 2380r^4 - \end{aligned}$$

$$\begin{aligned}
 & -2415r^6 + 1260r^8 - 350r^{10} + 48r^{12} \cos 2\theta + \\
 & + \frac{\mathcal{M}^2 \mathcal{R} \mathbb{E}}{48 \cdot 96 \cdot 240} (-19r^3 + 45r^5 - 36r^7 + 10r^9) \sin 3\theta + \\
 & + \frac{\mathcal{R}^2}{96 \cdot 96 \cdot 48 \cdot 120} (-37 + 180r^2 - 360r^4 + 380r^6 - 225r^8 + 72r^{10} - 10r^{12}) + \\
 & + \frac{1}{96 \cdot 12} (-1 + 3r^2 - 3r^4 + r^6) + \frac{1}{48 \cdot 96} (5r^2 - 8r^4 + 3r^6) \cos 2\theta \Big], \\
 (3) \quad {}_1\Phi = & -r \sin \theta + \mathcal{T}^2 \left[\frac{\mathbb{E}}{1152} (1 - 3r^2 + r^4 - r^6) - \right. \\
 & \left. - \frac{\mathbb{E}}{4608} (6r^2 - 8r^4 + 3r^6) \cos 2\theta \right],
 \end{aligned}$$

$$(4) \quad {}_2\Phi = {}_3\Phi = -r \sin \theta - \mathcal{T}^2 \left[\frac{\mathbb{E}}{4608} \frac{\cos 2\theta}{r^2} \right],$$

$$\begin{aligned}
 (5) \quad {}_1B_z = & \mathcal{T} \left[\left(A_1 r + \frac{\mathbb{E}}{4} r^3 \right) \cos \theta \right] + \mathcal{T}^2 \left[\left(pr + \frac{\alpha_1 r^3}{8} + \frac{\alpha_2 r^5}{24} \right) \cos \theta + \right. \\
 & \left. + \left(Lr^3 + \frac{\alpha_2 r^5}{144} \right) \cos 3\theta + \left\{ p_s r^2 + \frac{\alpha_3}{480} (10r^4 - 5r^6 + r^8) \right\} \sin 2\theta \right],
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad {}_2B_z = & \mathcal{T} \left[\left(A_2 r + \frac{A_3}{r} \right) \cos \theta \right] + \mathcal{T}^2 \left[\left(p^1 r + \frac{q^1}{r} \right) \cos \theta + \right. \\
 & \left. + \left(L^1 r + \frac{M^1}{r^3} \right) \cos 3\theta + \left(p_s^1 r^2 + \frac{q_s^1}{r^2} \right) \sin 2\theta \right],
 \end{aligned}$$

$$(7) \quad {}_3B_z = 0,$$

where

$$A_1 = -\frac{\mathbb{E}}{4} \left[\frac{3\sigma_2(\lambda^2 - 1) + \sigma_1(\lambda^2 + 1)}{\sigma_1(\lambda^2 + 1) + \sigma_2(\lambda^2 - 1)} \right],$$

$$A_2 = \frac{\mathbb{E} \sigma_2}{[\sigma_1(\lambda^2 + 1) + \sigma_2(\lambda^2 - 1)]},$$

$$A_3 = -\frac{\mathbb{E}}{2} \frac{\lambda^2 \sigma_2}{[\sigma_1(1 + \lambda^2) + \sigma_2(\lambda^2 - 1)]},$$

$$\alpha_1 = \left[\frac{\mathbb{E} A_1 \mathcal{M}^2}{2} - \frac{\mathbb{E}^2 \mathcal{M}^2}{24} \right],$$

$$\alpha_2 = \frac{3}{16} \in \mathcal{M}^2, \quad \alpha_3 = \frac{\varepsilon \mathcal{R}}{96},$$

$$p = - \frac{\sigma_2 (\lambda^2 - 1) (9\alpha_1 + 5\alpha_2) + \sigma_1 (\lambda^2 + 1) (3\alpha_1 + \alpha_2)}{24[\sigma_1(\lambda^2 + 1) + \sigma_2(\lambda^2 - 1)]},$$

$$p^1 = - \frac{\sigma_2(3\alpha_1 + 2\alpha_2)}{12[\sigma_1(\lambda^2 + 1) + \sigma_2(\lambda^2 - 1)]}, \quad q^1 = - \frac{\lambda^2 \sigma_2(3\alpha_1 + 2\alpha_2)}{12[\sigma_1(\lambda^2 + 1) + \sigma_2(\lambda^2 - 1)]},$$

$$L = - \frac{5\sigma_2\alpha_2(\lambda^6 - 1) + 3\sigma_1\alpha_2(1 + \lambda^6)}{432[\sigma_1(1 + \lambda^6) + \sigma_2(\lambda^6 - 1)]},$$

$$L^1 = \frac{\alpha_2\sigma_2}{216[\sigma_1(1 + \lambda^6) + \sigma_2(\lambda^6 - 1)]}, \quad M^1 = - \frac{\alpha_2\sigma_2\lambda^6}{216[\sigma_1(1 + \lambda^6) + \sigma_2(\lambda^6 - 1)]},$$

$$p_s = - \frac{\alpha_3}{160} \frac{3\sigma_2(\lambda^4 - 1) + 2\sigma_1(\lambda^4 + 1)}{\sigma_1(1 + \lambda^4) + \sigma_2(\lambda^4 - 1)},$$

$$p_s^1 = \frac{\alpha_3\sigma_2}{160[\sigma_1(1 + \lambda^4) + \sigma_2(\lambda^4 - 1)]}, \quad q_s^1 = - \frac{\alpha_3\sigma_2\lambda^4}{160\sigma_1(1 + \lambda^4) + \sigma_2(\lambda^4 - 1)}.$$

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CURGEREA HIDROMAGNETICĂ PRINTR-O CONDUCTĂ ÎN ROTAȚIE (II)

(Rezumat)

Se studiază problema curgerii hidromagnetice printr-o conductă circulară în rotație în jurul unei axe perpendiculară pe axa conductei. Cîmpul extern aplicat se presupune paralel cu axa de rotație. Se stabilește că efectul combinat al cîmpului magnetic și rotației reduce fluxul de masă prin conductă.

ON ELASTIC STABILITY IN THE CASE OF CONSERVATIVE LOADS

BY

ELISABETA RUSU

1. In [1] Koiter has developed an energy criterion of elastic stability, including the important nonlinear phenomena, the start point being Lyapunov's remarks on stability.

His basic assumption is that the external loads being conservative, it must exist a potential energy $P_1[u(P, t)]$ of the displacement field $u(P, t)$, whose time derivative in any actual or virtual motion be the negative of the rate of work of the external loads applied on the body.

In order to explicit Koiter's theory, an investigation of potential surface loads was needed; this been made by Sewell [3].

In this paper we are connecting the results of the above mentioned authors and we give a fully statement of Koiter's energy criterion for elastic stability in the questioned case. In the particular case of „dead loadings” we obtain a proof of the sufficient condition of stability, stated without proof in [5] and [7], by using Lyapunov's theory of stability. Finally, we get the condition of elastic stability in the case of uniforme potential loading, as obtained by Sewell [3].

2. We consider an elastic body acted by external loads, in an equilibrium state which we shall call the fundamental state.

Let be D domain of the space occupied by the medium whose boundary is Σ (a smooth surface), n is the unit outward normal to Σ . Σ_α ($\alpha = 1, 2$) denote subsets of Σ , thus $\Sigma = \Sigma_1 \cup \Sigma_2$. Suppose that Σ_1 is bounded by a closed smooth curve C .

We have the boundary conditions:

$$(1) \quad t = f \quad \text{on} \quad \Sigma_1, \quad u = u_0 \quad \text{on} \quad \Sigma_2,$$

where f and u_0 are prescribed functions. Let be F the body force.

In this fundamental state, the temperature is necessarily uniform at the value T_1 of the surroundings temperature.

A motion of the body, starting from the fundamental state is specified by the displacement field $\mathbf{u}(P, t)$ and a temperature field $T(P, t)$, both (vectors and respectively scalar) functions of position and time. Here P is an arbitrary point of the body, $P \in \bar{D} = D + \Sigma$.

Suppose that $\mathbf{u}$, together with its first derivatives, are continuous functions in respect to P and t .

Let be a system of curvilinear coordinates θ^i , the covariant and contravariant base vectors being $\mathbf{g}_i$ and $\mathbf{g}^i$ respectively.

Let

$$(2) \quad \gamma_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u^r_{,i} u_{r,j})$$

be the components of the strain tensor, where the comma means covariant derivative; we use the metric tensor $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$

$$u^j_{,i} = \mathbf{g}^j \frac{\partial \mathbf{u}}{\partial \theta^i}.$$

We have

$$(3) \quad F(\gamma_{ij}, T) = U(\gamma_{ij}, S) - TS, \quad S = - \frac{\partial F(\gamma_{ij}, T)}{\partial T},$$

where F is the Helmholtz's free energy function per unit mass, U — the internal energy per unit mass and S — the entropy per unit mass.

The kinetic energy

$$(4) \quad K = \frac{1}{2} \int_D \rho \dot{\mathbf{u}}^2 dV$$

is expressed in terms of velocity field $\dot{\mathbf{u}}(P, t)$; ρ is the mass density.

We assume that the specific heat c_γ at constant deformation and the absolute temperature T are both positive:

$$(5) \quad c_\gamma = - \frac{\partial^2 F}{\partial T^2} T > 0 \quad \text{and} \quad T > 0.$$

We postulate:

a) The equation of entropy balance:

$$(6) \quad \frac{d}{dt} \int_D S \rho dV = - \int_\Sigma \frac{\mathbf{q} \cdot \mathbf{n}}{T} d\Sigma + \int_D \sigma \rho dV,$$

where $\mathbf{q}(P, t)$ is a heat flow vector, $\sigma(P, t)$ is the internal entropy production rate per unit mass; the inequality of Clausius-Duhem is expressed by:

$$(7) \quad \sigma(P, t) \geq 0.$$

An increase of the body temperature T at the surface is then accompanied by an outward heat flow. We have therefore the inequality at the body surface

$$(8) \quad (T - T_1) \mathbf{q} \cdot \mathbf{n} \geq 0.$$

b) The equation of energy balance :

$$(9) \quad \frac{d}{dt} \int_D U(\gamma_{ij}, S) \rho dV + \int_D \rho \dot{\mathbf{u}}^2 dV = \int_{\Sigma} \mathbf{f} \dot{\mathbf{u}} d\Sigma + \int_D \mathbf{F} \dot{\mathbf{u}} \rho dV - \int_{\Sigma} \mathbf{q} \mathbf{n} d\Sigma.$$

Suppose that there is a potential energy V so that

$$(10) \quad \dot{V} = - \int_D \mathbf{F} \dot{\mathbf{u}} \rho dV,$$

where $\mathbf{F}$ is the body force (conservative [6]).

We assume that the surface force $\mathbf{f}$ is conservative in the sense of Sewell's existence theorem of the surface potential [3], [4].

Let be f in (1) having a specified functional form

$$(11) \quad \mathbf{f} = \mathbf{f}(\theta^i, u^i, u_p^j); \quad \mathbf{f} = f_i \mathbf{g}^i.$$

We require that f_i must be homogeneous functions, of degrees r and s in u^j and u_p^j respectively, where u_i are the components of displacement field $\mathbf{u} = u_i \mathbf{g}^i$ and u_p^j are determined by the following decomposition :

$$(12) \quad \begin{cases} u_{,i}^j = n_i \lambda^j + u_i^j, \\ \lambda^j = \mathbf{g}^j \frac{\partial \bar{u}}{\partial n} = u^{j,i} n^i \quad u_i^j = (\delta_i^k - n_i n^k) u_{,k}^j. \end{cases}$$

We see that f_i depends only of the surface value of $\mathbf{u}$ and its surface derivative, u_i^j (surface information).

There is a surface potential

$$(13) \quad \mathfrak{D} = \mathfrak{D}(\theta^i, u^i, u_p^j) = \frac{\mathbf{u} \cdot \mathbf{f}}{r + s + 1},$$

which is a homogeneous function of u^i and u_p^j with degrees $r+1$ and s respectively, depending only on the surface information; then we have

$$(14) \quad \frac{d}{dt} \int_{\Sigma} \mathfrak{D} d\Sigma = \frac{d}{dt} \int_{\Sigma} \frac{\mathbf{u} \cdot \mathbf{f}}{r + s + 1} d\Sigma = \int_{\Sigma} \mathbf{f} \cdot \dot{\mathbf{u}} d\Sigma + \oint_C \frac{\mathfrak{E}_{ijk}}{r + s + 1} n^j \frac{\partial f_p}{\partial u_{,k}^q} u^p u^q ds^i,$$

($ds^i = \tau^i ds$, τ is the unit tangent vector to C), if and only if, the following conditions hold :

$$(15) \quad \frac{\delta_k^i - n_k n^i}{r + s + 1} \left[(\delta_p^k - n_p n^k) u^j u^r \frac{\partial f_r}{\partial u_p^j} \right]_{,i} = 0,$$

$$u^k \left[\frac{\partial f_j}{\partial u_k} - \frac{\partial f_k}{\partial u_j} \right] + u_p^r \left[\frac{\partial f_j}{\partial u_p^r} + \frac{\partial f_r}{\partial u_p^j} \right] + (\delta_k^i - n_k n^i) \left[(\delta_p^k - n_p n^k) \frac{\partial f_r}{\partial u_p^j} \right]_{,i} u^r = 0.$$

It follows that by (14) $\int_{\Sigma} \mathbf{f} \cdot \dot{\mathbf{u}} d\Sigma$ is derivable from a surface potential when the line integral vanishes. This happens in particular if

$$(16) \quad \begin{cases} \text{(i)} \quad \frac{\partial f_r}{\partial u_{,k}^j} = 0, & \text{(ii)} \quad \dot{\mathbf{u}} = 0 \text{ on the path,} \\ \text{(iii) the path has zero length, as } \Sigma_1 \text{ is a closed surface.} \end{cases}$$

The surface loads $\mathbf{f}$ is then called conservative [3], [4] and by (14) we have :

$$(17) \quad \frac{d}{dt} \int_{\Sigma} \mathfrak{D} d\Sigma = \frac{d}{dt} \int_{\Sigma} \frac{\mathbf{u} \cdot \mathbf{f}}{r + s + 1} d\Sigma = \int_{\Sigma} \mathbf{f} \cdot \dot{\mathbf{u}} d\Sigma.$$

Suppose that (10) and (17) holds. Now (9) may be written in the form :

$$(18) \quad \frac{d}{dt} \left\{ \int_D [U(\gamma_{ij}, S) - T_1 S] \rho dV + \int_D \rho \dot{\mathbf{u}}^2 dV + \mathfrak{E} - \int_{\Sigma} \mathfrak{D} d\Sigma \right\} = - \int_{\Sigma} \mathbf{q} \cdot \mathbf{n} d\Sigma.$$

We follow Koiter [1] in order to introduce a Lyapunov function.

We reduce this equation, by subtracting the entropy balance equation (6) multiplied by T_1 , to

$$(19) \quad \frac{d}{dt} \left\{ \int_D [U(\gamma_{ij}, S) - T_1 S] \rho dV + \int_D \dot{\mathbf{u}}^2 \rho dV + \mathfrak{E} - \int_{\Sigma} \mathfrak{D} d\Sigma \right\} =$$

$$= \int_D \left(\frac{T_1}{T} - 1 \right) \mathbf{q} \cdot \mathbf{n} d\Sigma - T_1 \int_D \rho \sigma dV \leq 0,$$

where the right-hand member is always nonpositive in virtue of the inequalities (7) and (8).

The first term in left hand may be written

$$(20) \quad U(\gamma_{ij}, S) - T_1 S = U(\gamma_{ij}, S) - TS + TS - T_1 S =$$

$$= F(\gamma_{ij}, S) - (T_1 - T) \frac{\partial F(\gamma_{ij}, T)}{\partial T}.$$

Now we employ the Taylor expansion of the free energy in the vicinity of the current temperature T :

$$(21) \quad F(\gamma_{ij}, T_1) = F[\gamma_{ij}, T + (T_1 - T)] = F(\gamma_{ij}, T) + (T_1 - T) \frac{\partial F(\gamma_{ij}, T)}{\partial T} + \frac{1}{2!} (T_1 - T)^2 \frac{\partial^2 F^*}{\partial T^{*2}},$$

where

$$T^* = T + \lambda(T_1 - T), \quad 0 < \lambda < 1, \quad -\frac{\partial^2 F^*}{\partial T^{*2}} T^* = c_\gamma^* > 0.$$

By (20) and (21) we have:

$$(22) \quad U(\gamma_{ij}, S) - T_1 S = F(\gamma_{ij}, T_1) + \frac{1}{2} (T_1 - T)^2 \frac{c_\gamma^*}{T^*}.$$

The balance equation (19) is not affected by subtraction from the expression between braces in the left hand member, the time independent equality:

$$(23) \quad \int_D \rho [U(0, S_1) - S_1 T_1] dV = \int_D F(0, T_1) \rho dV,$$

where S_1 is the entropy in the fundamentele state.

In the expression

$$(24) \quad F(\gamma_{ij}, T_1) - F(0, T_1) = W(\gamma_{ij}, T_1)$$

we recognise the (additional) stored elastic energy, per unit mass, in the fundamental state in an isothermal (additional) deformation γ_{ij} , at constant temperature T_1 .

We may write now (19) in the form:

$$(25) \quad \frac{d}{dt} \left\{ \int_D W(\gamma_{ij}, T_1) \rho dV + \int_D \dot{\mathbf{u}}^2 \rho dV + \int_{\Sigma} \mathfrak{D} d\Sigma + \frac{1}{2} \int_D \frac{c_\gamma^*}{T^*} (T_1 - T)^2 \rho dV \right\} \leq 0.$$

Let denote

$$(26) \quad \mathfrak{E} = \int_D W(\gamma_{ij}, T_1) \rho dV + \int_{\Sigma} \mathfrak{D} d\Sigma,$$

which is the total potential energy and

$$(27) \quad \mathfrak{D} = \mathfrak{E} + \int_D \dot{\mathbf{u}}^2 \rho dV + \frac{1}{2} \int_D \frac{c_\gamma^*}{T^*} (T_1 - T)^2 \rho dV,$$

called by D u h e m, the ballistic energy.

3. Definition [1], [2]. The equilibrium configuration in the fundamental state is called stable in the sens of Lyapunov, if and only if, the body remains always „close” to these equilibrium configuration in the motion following upon an arbitrary „sufficiently small” initial disturbance.

For a precise definition, we need quantitative measures for the distance to the equilibrium configuration and for the magnitude of the initial disturbance.

We define the norms at time t , in L_2 , by the equations :

$$(28) \quad \begin{aligned} \| \mathbf{u} \|_t^2 &= \frac{1}{M} \int_D \rho \mathbf{u} \cdot \mathbf{u} \, dV, & \| \dot{\mathbf{u}} \|_t^2 &= \frac{1}{M} \int_D \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dV, \\ \| T_1 - T \|_t^2 &= \frac{1}{M} \int_D \rho (T_1 - T)^2 \, dV, \end{aligned}$$

where M is the body total mass.

We shall assume that the potential energy $\mathfrak{Q}$ defined by (27) is regular in the following sense : on every sphere $\| \mathbf{u} \| = c$ in the space of kinematically admissible displacement fields $\mathbf{u}$, when the radius c is sufficiently small, the energy $\mathfrak{Q}$ has a proper minimum, which is a continuous function $d(c)$, in the range of c under consideration.

The magnitude of an arbitrary initial disturbance is measured in terms of L_2 -norm of the displacement field at $t = 0$, and the initial value $\mathfrak{Q}_0$, at $t = 0$, of the ballistic energy.

Koiter's precise definition of stability may be stated in the following terms : The equilibrium in the fundamental state will be called stable, if and only if for every triple of positive constants α_1 , α_2 and α_3 a pair of positive constant β_1 and β_2 exists, such that in every motion following after a disturbance at time $t = 0$, whose magnitude is restricted by

$$(29) \quad \| \mathbf{u} \|_0 < \beta_1, \quad \mathfrak{Q}_0 < \beta_2,$$

we have for the distance to the fundamental state for all positive values of time,

$$(30) \quad \| \mathbf{u} \|_t < \alpha_1, \quad \| \dot{\mathbf{u}} \|_t < \alpha_2, \quad \| T_1 - T \|_t < \alpha_3.$$

In fact, if E^* is a positive lower bound of c_T^*/T^* , it is easily verified that the positive bounds in (29)

$$\beta_1 = \alpha_1 \quad \text{and} \quad \beta_2 = \min \left\{ d(\alpha_1), \frac{1}{2} M \alpha_2^2, \frac{1}{2} E^* M \alpha_3 \right\}$$

ensure the inequalities (30) for all positive values of time.

An immediate consequence of Lyapunov general stability theorem is that

$$(31) \quad \mathcal{E} = \int_D W(\gamma_{ij}, T_1) \rho dV + \mathcal{V} - \int_{\Sigma} \mathcal{D} d\Sigma > 0$$

is a sufficient condition for stability.

The sufficient condition of stability of equilibrium of an elastic body subject to a system of external loads (body force and surface force) conservative in the foregoing sense, is that in any possible displacement from the position of the equilibrium, the strain energy stored or dissipated should exceed potentials of body force and surface loads.

4. „Dead” loadings. Several authors [5 p. 276], [6, p. 599], [7, p. 133] have defined the dead loadings: the body force $\mathbf{F}$ and the assigned surface force $\mathbf{f} d\Sigma$ are the same as in the corresponding points on the actual motion, and we have

$$(32) \quad \frac{d}{dt} \left(- \int_{\Sigma} \mathcal{D} d\Sigma + \mathcal{V} \right) = \int_{\Sigma} \mathbf{u} \cdot \mathbf{f} d\Sigma + \int_D \mathbf{F} \cdot \mathbf{u} \rho dV.$$

We also may obtain the first term of right hand, by the surface potential (13) for the particular case when $\mathbf{f}$ is a homogeneous function of degrees $r=0$ and $s=0$ in u^j and u_p^j respectively

$$(33) \quad \mathcal{D} = \frac{\mathbf{u} \cdot \mathbf{f}}{r + s + 1} = \mathbf{u} \cdot \mathbf{f}.$$

In this case, the sufficient condition of stability becomes that given without proof in [5] and [7]: „A sufficient condition for stability is that in any possible displacement from the position of equilibrium, the strain energy stored or dissipated, should exceed the work done on the system by external loads (body force and surface force)“.

We have obtained here the proof of this sufficient condition for stability, starting from Lyapunov's definition of stability.

5. *Uniform pressure loading.* We shall obtain the expression of the condition for elastic stability (31) in the case of an elastic body subject to an uniform pressure loading. In this purpose we employ the expression of surface potential obtained by Sewell [3].

Let be Σ the body surface in the fundamental state. Suppose that in the motion following the displacement $\mathbf{u}$, and taking place in a time τ , the body has a new position, with Σ_τ surface.

Let θ^i be the general system of curvilinear coordinates. We define another system of coordinates, ζ_i^j by representing the position vector $\mathbf{p}$ of the points in the neighbourhood of Σ by

$$(34) \quad \mathbf{p}(\zeta^1, \zeta^2, \zeta^3) = \mathbf{r}(\zeta^1, \zeta^2) + \zeta^3 \mathbf{n}(\zeta^1, \zeta^2).$$

The equation of Σ surface is

$$\zeta^3 = 0 \quad \text{or} \quad h(\theta^1, \theta^2, \theta^3) = 0$$

for all τ , and ζ^1, ζ^2 are surface coordinates,

Let the original vector area element $\mathbf{n}d\Sigma$ (in the fundamental state) become now $\mathbf{N}d\bar{\Sigma}$ (say), where $\mathbf{N}$ is the new unit normal to the surface and $d\bar{\Sigma}$ the new value of the area element. Then

$$(35) \quad \mathbf{N}d\bar{\Sigma} = \frac{\delta(\mathbf{r} + \mathbf{u})}{\delta\zeta^1} \times \frac{\delta(\mathbf{r} + \mathbf{u})}{\delta\zeta^2} d\zeta^1 d\zeta^2,$$

whence

$$(36) \quad \mathbf{N} \frac{d\bar{\Sigma}}{d\Sigma} = \left[n_k + (n_k u_{,j}^i - n_j u_{,k}^i) + \frac{1}{2} \epsilon_{ijk} \in^{pqr} n_r u_{,p}^i u_{,q}^j \right] g^k.$$

The covariant derivatives $u_{,i}^j = g^j \frac{\partial u}{\partial \theta_i}$ are here evaluated still using the original base vectors g^j on Σ . That (36) contains only surface data is also easily checked from (12) by verifying the absence of $\bar{\lambda}$. The coefficient of g^k in (36) contains three terms which are homogeneous of degree zero, one and two in the displacement surface gradients, and is independent of $\mathbf{u}$ itself. We verify that each term satisfies (15), hence (14) apply for each term.

Suppose now that uniform pressure loading, of magnitude p per unit area is applied to Σ_1 throughout the deformation.

That is, the load vector

$$(37) \quad -p\mathbf{N}d\bar{\Sigma} = \mathbf{f}d\Sigma$$

(thus defining $\mathbf{f}$) is specified to have the functional form defined by:

$$(38) \quad f_k = -p \left[n_k + (n_k u_{,j}^i - n_j u_{,k}^i) + \frac{1}{2} \epsilon_{ijk} \in^{pqr} n_r u_{,p}^i u_{,q}^j \right],$$

in which p is independent of position.

Since then the line integral in (14) vanishes, we have [3]:

$$(39) \quad \int_{\Sigma} p \mathbf{N} d\bar{\Sigma} = \frac{d}{dt} \left[p \left\{ \int_{\Sigma} n_z u^k d\Sigma + \frac{1}{2} \int_{\Sigma} (n_k u_{,j}^i - n_j u_{,k}^i) u^k d\Sigma + \right. \right. \\ \left. \left. + \frac{1}{6} \int_{\Sigma} \epsilon_{ijk} \in^{pqr} u^k n_r u_{,p}^i u_{,q}^j d\Sigma \right\} \right].$$

The condition of stability may be written in the form:

$$(40) \quad \int_D W(\gamma_{ij}, T_1) dV - p \left[\int_{\Sigma} n_k u^k d\Sigma + \frac{1}{2} \int_{\Sigma} (n_k u_{,j}^i - n_j u_{,k}^i) u^k d\Sigma + \right. \\ \left. + \frac{1}{6} \int_{\Sigma} \epsilon_{ijk} \in^{pqr} u^k n_r u_{,p}^i u_{,q}^j d\Sigma + \infty > 0 \right].$$

It is the same condition as in [4], but there it was obtained using another definition of the elastic stability.

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ASUPRA STABILITĂȚII ELASTICE ÎN CAZUL ÎNCĂRCĂRILOR
CONSERVATIVE

(Rezumat)

Se stabilește o formă explicită a criteriului lui Koiter [1] de stabilitate a echilibrului unui corp deformat, utilizând în acest scop potențialul forțelor superficiale găsit de Sewell [3].

Ca aplicații, se studiază pe baza teoriei lui Liapunov cazul încărcărilor moarte, obținând o demonstrație a criteriului suficient de stabilitate enunțat fără demonstrație de exemplu în [5] și [7], și cazul încărcării datorită presiunii hidrostactice uniforme, pentru care potențialul a fost găsit de Sewell [3].

RECHERCHES MÉTALLOGRAPHIQUES CONCERNANT LA STRUCTURE DES COMPOSÉS DU SYSTÈME $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ ¹⁾

PAR

ANASTASIA VASILIU, VLAD CEHAN, C. IGNAT et EMIL LUCA

Les recherches expérimentales effectuées sur les composés du système $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ ont mis en évidence une relation étroite entre la structure de ceux-ci, déterminée par des diffractiions aux rayons X [1] et les propriétés électriques et magnétiques comme : conductivité, résistivité et énergie d'activation ; permittivité et pertes diélectriques ; perméabilité et température Curie ; magnétisation de saturation et pertes par histéresis tournant.

Dans le but de compléter ces études, nous nous proposons d'effectuer aussi certaines recherches métallographiques concernant la structure des composés du système $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ dans le but d'interpréter certaines des propriétés électriques de ceux-ci.

Les mesures expérimentales effectuées à l'aide d'un microscope métallographique Neophot, ont compris les ferrites du tableau 1, en pour-

Tableau 1

Ferrite	x	Ni %	Ca %	Fe %	O %
<i>I</i>	0	25,05	0	47,65	27,30
<i>II</i>	0,05	23,89	0,86	47,84	27,41
<i>III</i>	0,10	22,73	1,72	48,03	27,52
<i>IV</i>	0,15	21,55	2,59	48,23	27,63
<i>V</i>	0,20	20,36	3,48	48,42	27,74
<i>VI</i>	0,30	17,95	5,25	48,82	27,97
<i>VII</i>	0,40	15,52	7,06	49,22	28,20
<i>VIII</i>	0,50	13,05	8,90	49,62	28,43
<i>IX</i>	0,70	7,96	12,67	50,46	28,81

¹⁾ Présentée lors de la session scientifique jubilaire „60 ans d'enseignement électrotechnique à Jassy“, le 21—23 décembre 1972.

suivant, en particulier, la variation de la porosité de ceux-ci, fonction du contenu en ions de Ca, ce qui a imposé l'utilisation de certaines preu es aux faces non-attaquées.

Des microphotos obtenues à une puissance d'augmentation de 100, nous présentons celles des fig. 1, 2 et 3 qui se rapportent à la ferrite NiFe_2O_4 et aux composés $\text{Ca}_{0,2}\text{Ni}_{0,8}\text{Fe}_2\text{O}_4$ et $\text{Ca}_{0,7}\text{Ni}_{0,3}\text{Fe}_2\text{O}_4$, dans laquelle une partie des ions de Ni, du réseau cristallin, ont été substitués aux ions de Ca.

De l'analyse des microphotos, il en résulte :

a) La ferrite de NiFe_2O_4 présente des micropores de dimensions réduites, répartis d'une façon uniforme dans la masse fondamentale, leur provenance étant due au processus de pression et syntérisation.

b) Par la substitution de certains des ions de Ni par des ions de Ca, la structure de la ferrite obtenue se modifie. Le composé $\text{Ca}_{0,2}\text{Ni}_{0,8}\text{Fe}_2\text{O}_4$ (fig. 2) présente une augmentation évidente des micropores formés dont les dimensions sont différentes, ayant la tendance de concentration et sphéroïdisation et étant reparti d'une manière non uniforme dans la masse fondamentale du composé.

c) Les dimensions et le degré de non-uniformité et distribution dans la masse fondamentale des micropores formés augmente en même temps que l'augmentation en ions de Ca des composés du système $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$. Ainsi, au composé $\text{Ca}_{0,7}\text{Ni}_{0,3}\text{Fe}_2\text{O}_4$ (fig. 3) les pores formés, de dimensions beaucoup plus grandes, sont élargis sous forme insulaire, en présentant un aplatissement et une division en couches en profondeur, ce qui diminue notamment l'homogénéité du composé.

Tout cela met en évidence une relation étroite entre la forme, les dimensions et la distribution des pores et la concentration en ions de Ca des composés du système $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$.

Dans le but d'obtenir des données plus complètes concernant la géométrie des pores formés on a effectué la microphotographie de la surface du composé $\text{Ca}_{0,7}\text{Ni}_{0,3}\text{Fe}_2\text{O}_4$ à des puissances d'augmentation plus grandes, en obtenant les photos présentées dans les fig. 4, 5 et 6.

Dans la fig. 4 il est reproduit, augmenté 1 200 fois, un pore de la surface du composé $\text{Ca}_{0,7}\text{Ni}_{0,3}\text{Fe}_2\text{O}_4$. On observe clairement l'apparition dans le volume de celui-ci des sectionnements des cristaux qui ont poussé les plans antérieurs, sans contact avec la masse fondamentale du plan de sectionnement utilisé.

Pour obtenir des données supplémentaires concernant, en particulier, la nature des frontières entre la cristalite sectionnée et la masse fondamentale on a effectué la microphotographie du micropore à une puissance d'augmentation de 1 800, représenté dans la fig. 5; ensuite, à la même puissance d'augmentation, on analyse uniquement la zone confuse du micropore, présentée dans la fig. 6.

À la base de la cristalite sectionnée on peut remarquer des cristaux en train de se former aux frontières interrompues, comme résultat du fait qu'on n'a pas assuré de contact mécanique complet des poussières durant

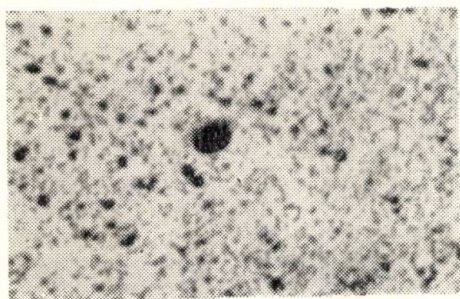


Fig. 1

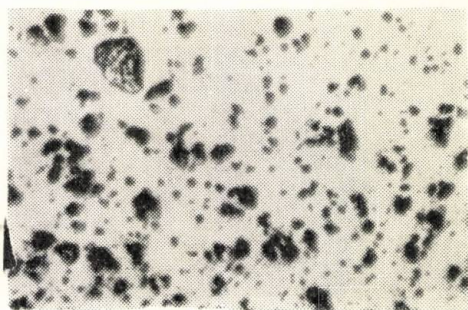


Fig. 2

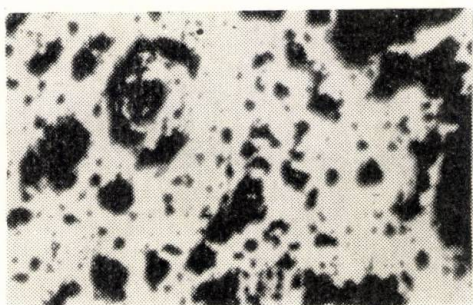


Fig. 3



Fig. 4

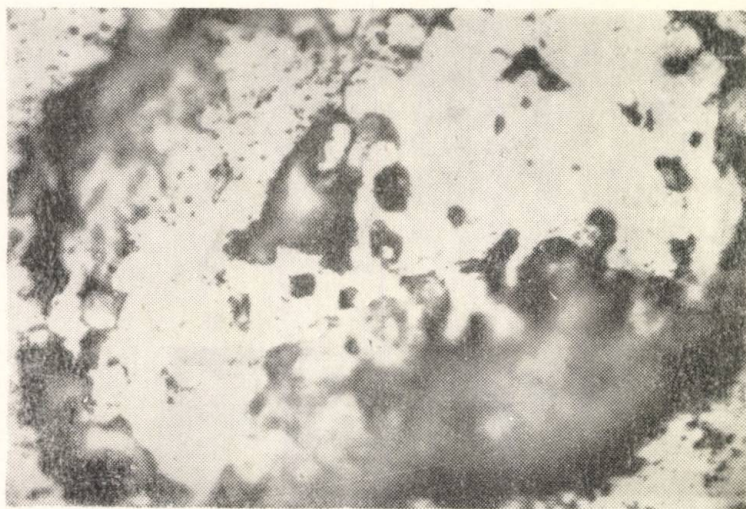


Fig. 5

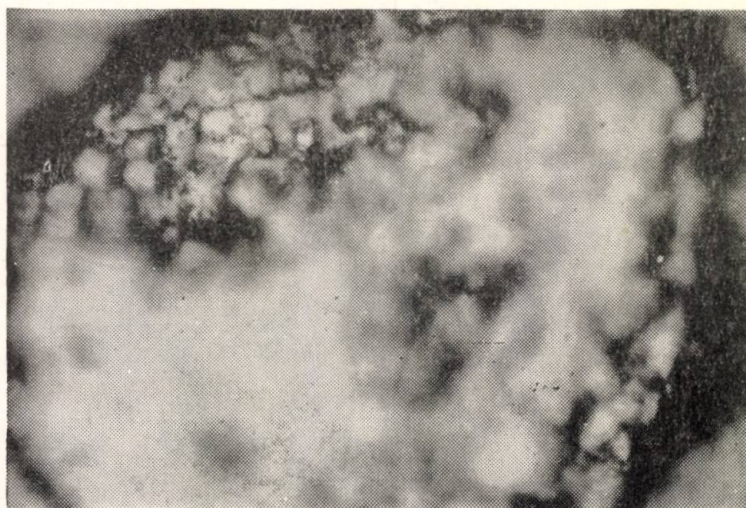


Fig. 6

la pression des preuves, du manque des ions de Ca pendant la syntérisation et des contractions des épreuves pendant le processus de refroidissement.

Les recherches métallographiques effectuées ont mis en évidence aussi le fait que le processus d'augmentation des crystalites formées a lieu dans une direction perpendiculaire par rapport au plan de sectionnement des preuves, donc parallèlement à la direction de tensionnement durant la formation de la preuve et que grâce à l'existence de ces tensions mécaniques apparaissent des fissures dirigées le long des mêmes directions.

Le processus de formation et évolution des pores dépendant de la concentration en ions de Ca au composé $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ nous le considérons comme étant déterminé par plusieurs facteurs comme : le volume de l'atome de Ca qui est quatre fois plus grand que celui de l'atome de Ni, qu'il substitue dans le réseau cristallin ; la température de fusion du Ca, qui est de beaucoup inférieure à la température de syntérisation ; aux tensions mécaniques durant la préparation des preuves et aux processus de contraction pendant le refroidissement, lent ou rapide de celles-ci.

Conformément à ceux qu'on a signalé [1] ... [6], par l'addition de ions de Ca dans la ferrite NiFe_2O_4 , un mélange de phases en naît. En tenant compte de la grande affinité du Ca par rapport à l'oxygène, nous considérons qu'à des petites additions de Ca, les oxydes CaO et CaO_2 sont présents.

À des petites concentrations de Ca, les oxydes formés, eux aussi en faible quantité, sont insuffisants pour former des frontières aux crystalites et de cette façon des couches isolatrices. Mais grâce à la réduction de l'oxygène de la ferrite fondamentale NiFe_2O_4 on crée la possibilité de formation de ponts de ferrite continus qui augmentent la conductibilité du composé, ce qui s'accorde aux résultats expérimentaux [2], [3].

Par l'augmentation de la concentration en ions de Ca, celui-ci se trouvant à l'état liquide, grâce à la température réduite de fusion, il sera poussé par les crystalites en formation vers les frontières ou il précipitera [7]...[10]. Il apparaît ainsi la possibilité de formation de couches isolatrices autour des crystalites qui seront d'autant plus complètes et plus denses que la concentration en ions de Ca sera plus grande, ce qui détermine des augmentations rapides de la résistivité [2], [3] et [7]...[10], à côté de l'augmentation de la résistivité produite par l'existence de plusieurs phases.

Le surplus de Ca non-précipité aux frontières des crystalites est poussé dans les vides formés par le processus de compression, en déterminant l'apparition et l'augmentation des crystalites.

Au refroidissement, grâce aux concentrations qui se produisent, les pores apparaissent, aussi bien que certaines fissures, déterminées aussi par l'existence d'un état tensionné. Celle-ci ont pour effet la réduction considérable de l'homogénéité et de la compacité du composé.

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CERCETĂRI METALOGRAFICE ASUPRA STRUCTURII
COMPUȘILOR SISTEMULUI $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$

(Rezumat)

Utilizînd probe ale sistemului $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$, de compoziție și moduri de preparare diferite, s-au efectuat unele studii metalografice, pe fețele neatacate, privind variația porozității și influența sa asupra rezistivității.

C. D. 537.311.33:678

SUR QUELQUES PROPRIÉTÉS DES COUCHES MINCES
POLYMÉRIQUES OBTENUS EN IMPULSION INITIALE ¹⁾

PAR

OVIDIU SCUTARU et VLAD CEHAN

Parmi les méthodes utilisées dans le procédé de préparation synthétique s'est imposée dernièrement. celle connue sous la dénomination de „polymérisation en plasma“ obtenue dans un champ de radiofréquence, avec des électrodes extérieurs, par couplage capacitif [1]...[7].

Les recherches dans ce domaine, liées spécialement aux méthodes expérimentales utilisées, apportent continuellement des éléments nouveaux, dont s'impose d'être cité le procédé d'alimentation contrôlée du plasma, avec des monomères, la vitesse d'alimentation étant déterminée par la rapidité des processus du plasma, ce qui mène à la polymérisation totale des vapeurs [1].

Par condensation du plasma froid en champ électrique de radiofréquence (polymérisation cathodique réactive de type diode) on peut obtenir des couches minces polymérique, avec des remarquables propriétés semi-conductrices [9], [10]. À l'aide du procédé expérimental qui utilise l'alimentation contrôlée du plasma [1]...[3], mais dans des conditions spéciales de travail, nous nous proposons d'obtenir des couches minces polymériques, auxquelles nous allons déterminer la conductivité, la résistivité et l'énergie d'activation, de même que leur variation avec la température et l'épaisseur des couches déposées.

Les processus de polymérisation ont été obtenues par impulsion initial du déchargement électrique luminescent, donc dans un état de transit où les vapeurs du monomère se trouvant à la température de l'ambiance (d'environ 20°C) éteignent le déchargement luminescent. On évite ainsi le

¹⁾ Présentée lors de la session scientifique jubilaire „60 ans d'enseignement électrotechnique à Jassy“, le 21—23 décembre 1972.

refroidissement du monomère pour que la tension de ses vapeurs saturés permette de maintenir le déchargement luminiscent dans le réacteur et même la refroidissement du monomère à -36°C .

La méthode demande cependant d'établir exactement les conditions du travail, au contraire les films n'étant pas uniformes et adhérents, parce que le produit se pulvérise dans le réacteur.

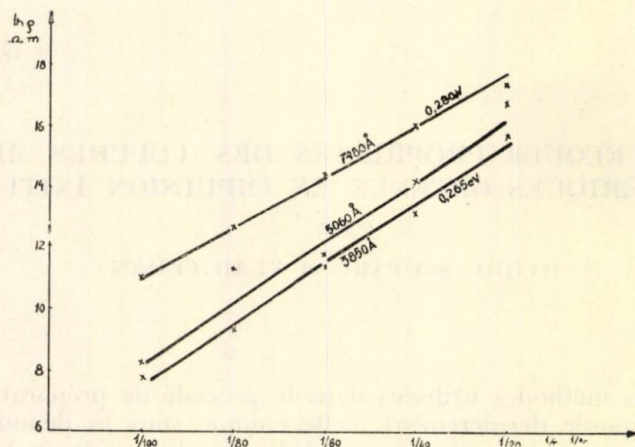


Fig. 1

La production du plasma a été réalisée, dans des conditions semblables à celle décrites dans la littérature, dans un réacteur cylindrique, en verre raso-thérmiq, inactif du point de vue chimique. Le champ électrique de haute fréquence (30 MHz) était obtenu à l'aide d'un générateur de radio-fréquence, avec un tube de GR-500 à une puissance de quelques centaines de Watts.

Après la réalisation du vide de $10^{-3} \dots 10^{-4}$ mm. col. Hg., évidencié par la disparition de la luminiscence et l'installation d'un vide noir, le réacteur avait été séparé de l'installation de vide et connecté à un réservoir de monomères (benzène) termo-réglé. La température du composant de départ détermine la pression des vapeurs saturés dans le réacteur.

En arrivant dans le réacteur, où a lieu le déchargement électrique luminiscent, crée par le champ électrique de haute fréquence, les vapeurs de benzène produisent l'extinction de ce déchargement à cause de la tension des vapeurs saturés, trop grande pour maintenir le déchargement.

Dans ce processus transitoire, d'extinction du plasma, a lieu l'apparition de la phase solide déposé en fine pellicule sur les parois du réacteur et sur les lamelles placées spécialement au paroi. Le produit est de couleur jaune jusqu'au brun; il est très adhésif aux parois du réacteur et difficilement soluble dans les solvants habituels.

Si la durée d'existence du plasma est petite, le film est formé d'une série de contours hexagonaux qui s'étendent jusqu'à l'enveloppement complet de la lamelle avec une couche homogène.

Parmi les propriétés électriques du polymère benzénique, on a étudié seulement la variation de la résistivité de la pellicule avec la température, à l'aide d'un électromètre ORION. L'épaisseur des couches mesurée

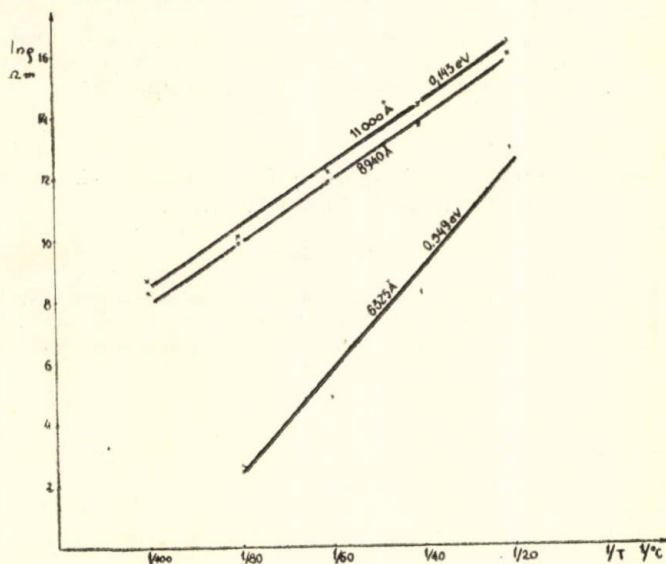


Fig. 2

avec un microinterferomètre type MII-4 est de l'ordre $5.10^{-7} \dots 11.10^{-7} \text{ m}$. Pour effectuer les mesures expérimentales on a réalisé des contacts d'argent colloïdale avec un liant.

Les courbes données dans les fig. 1 et 2 sont tracées pour deux groupes de produits, en prenant dans l'ordonnée le logarithme de la résistivité et en abscisse, l'inverse de la température absolue. L'allure des droites qui y résultent montre que les polymères obtenus de la manière proposée sont semi-conducteurs, ce qu'on peut vérifier aussi avec la valeur des énergies d'activation calculées à l'aide de ces droites.

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ASUPRA UNOR PROPRIETĂȚI ALE PĂTURILOR SUBȚIRI POLIMERICE OBTINUTE ÎN IMPULS ÎNȚIAL

(Rezumat)

În condiții speciale de lucru, utilizând metoda „plasmopolimerizării” se obțin unele pături subțiri polimerice în impuls inițial al descărcării. Polimerii posedă proprietăți semiconductoare, ca și produșii obținuți prin „plasmopolimerizare” obișnuită.

L'ANISOTROPIE INDUITE DANS LES FERRITES Li—Fe—Co ET L'INFLUENCE D'ORDONNANCE DES IONS SUR CELLE-CI

PAR

ELENA MOLDOVANU, GH. MAXIM et V. PETRESCU

Le ferrite de lithium $\text{Li}_{0,5}\text{Fe}_{2,5}\text{O}_4$ présente une ordonnance des cations dans le sous-réseau octaédrique, nommé ordonnance du type 1 : 3 [1].

Dans un travail antérieur [2], nous avons montré que l'ordonnance du type 1 : 3 du ferrite de lithium influence l'anisotropie induite par traitement thermique sous champ magnétique (TTM); avec l'augmentation du degré d'ordre à longue distance, l'anisotropie induite par l'ordre directionnel décroît.

Les impuretés de cobalt ont une forte influence sur la grandeur de l'anisotropie induite dans les ferrites, dans le domaine de température compris entre 100 et 400°C [3], [4]. Pour cela nous avons étudié l'anisotropie induite par traitement thermique sous champ magnétique dans les ferrites Li—Fe—Co et la manière dans laquelle celle-ci est influencée par l'ordonnance des ions de lithium. Dans ce travail nous présentons les résultats obtenus.

Les recherches ont été effectuées sur les ferrites polycristallins du système $\text{Li}_{0,5-y/2}\text{Co}_y\text{Fe}_{2,5-y/2}\text{O}_4$ avec $y = 0; 0,02; 0,05$ et 0,1, préparés par la technologie ordinaire des céramiques, en employant CO_3Li_2 , Co_2O_3 et de l' $\alpha\text{-Fe}_2\text{O}_3$ en poudre. Le frittage a été fait en présence de l'air, à 1150°C, pendant 4 heures.

Les échantillons en l'état complètement désordonné ont été obtenus par la trempe dans l'eau, à la température de frittage [5]. Les échantillons dans l'état ordonné ont été obtenus par refroidissement à une vitesse de 1°C/min., dans le domaine de température 800...380°C.

On a mesuré l'anisotropie induite dans le régime isothermique à l'aide d'un magnétomètre de torsion avec équilibration automatique [6]. Pour des mesures, on chauffait l'échantillon pendant approximativement 10 minutes jusqu'à la température de travail, $T = 380^\circ\text{C}$. Puis

on appliquait un champ magnétique de $4 \cdot 10^5$ A/m, le long d'une certaine direction. On déterminait la valeur de l'anisotropie à un moment donné, en tournant le champ magnétique avec 45° et en mesurant le couple de torsion immédiatement après la rotation. Après la mesure du couple de torsion, on tournait le champ dans la direction du traitement et on continuait le traitement.

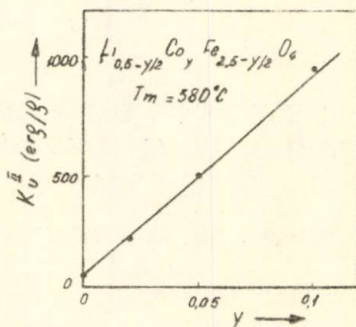


Fig. 1. — Variation de l'anisotropie induite K_u dans les ferrites $\text{Li}_{0.5-y/2}\text{Co}_y\text{Fe}_{2.5-y/2}\text{O}_4$ en fonction de la concentration des ions de cobalt y .

Dans la fig. 1 nous présentons la variation de l'anisotropie induite à $T = 380^\circ\text{C}$, mesurée en régime isothermique, aux ferrites $\text{Li}_{0.5-y/2}\text{Co}_y\text{Fe}_{2.5-y/2}\text{O}_4$ en fonction de la concentration des ions de cobalt (y). De l'étude de l'anisotropie induite aux ferrites Li—Fe—Co dans le domaine II de relaxation nous avons constaté que celle-ci présente deux composantes : a) la première, causée par l'ordre directionnel des ions de lithium (K_u^{Li} [7], ayant une origine semblable à l'anisotropie induite dans le même domaine de température aux ferrites Ni—Fe, Mn—Fe et Mg—Fe [4], [8] ; b) la seconde, due à l'ordre des ions uniques de cobalt sur les quatre catégories des sites octaédriques (K_u^{Co}), fait confirmé par la variation linéaire de l'anisotropie induite en fonction de la teneur de cobalt [4], [9].

Sur les ferrites Li—Fe—Co dans l'état ordonné et désordonné nous avons étudié la manière de variation de l'anisotropie induite en fonction de la durée du TTM à la température de 380°C . Pour l'échantillon ordonné, l'anisotropie induite augmente monotonement avec la durée de TTM, en tendant vers une valeur de saturation. Cette dépendance était à prévoir et elle est en concordance avec la théorie de Néel [10].

Pour l'échantillon désordonné l'anisotropie induite augmente avec la durée de TTM, touche un maximum K_u^{max} , la valeur duquel est plus grande que la valeur de l'anisotropie à l'échantillon ordonné ; puis elle décroît vers une valeur d'équilibre K_u^{ex} . Nous notons la décroissance de la valeur d'équilibre de l'anisotropie induite $K_u^{\text{max}} - K_u^{\text{ex}}$ par ΔK_u .

Dans la fig. 2 nous avons représenté la variation du rapport $K_u^{\max}/K_u^{\text{ex}}$ en fonction de la durée de TTM, pour le ferrite $\text{Li}_{0,45}\text{Fe}_{2,45}\text{Co}_{0,1}\text{O}_4$ ordonné (courbe 1) et désordonné (courbe 2). Au ferrite ordonné le rapport augmente monotonement jusqu'à la valeur de saturation (courbe 1). Au ferrite désordonné le rapport augmente au commencement, touche un maximum de valeur 1,3 puis décroît lentement avec une constante de

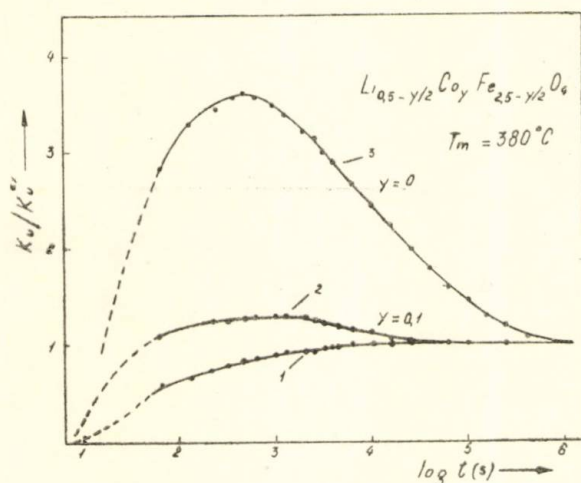


Fig. 2. — Variation du rapport K_u/K_u^{ex} au ferrite $\text{Li}_{0,45}\text{Co}_{0,1}\text{Fe}_{2,45}\text{O}_4$ en fonction de la durée de TTM à $T = 380^\circ\text{C}$; courbe 1 — échantillon ordonné; courbe 2 — échantillon désordonné; courbe 3 — ferrite $\text{Li}_{0,5}\text{Fe}_{2,5}\text{O}_4$, désordonné.

temps de $3,6 \cdot 10^3$ s vers la valeur d'équilibre. La courbe 3 représente la manière de variation du rapport K_u/K_u^{ex} en fonction de la durée de TTM pour le ferrite de lithium désordonné. On observe que la valeur maximum du rapport K_u/K_u^{ex} est beaucoup plus grande au ferrite de lithium.

Dans [2] nous avons montré que la décroissance de l'anisotropie induite avec la durée de TTM aux ferrites Li-Fe est causé par l'ordonnance à longue distance sur les sites octaédriques. On remarque de la comparaison des courbes 2 et 3 (fig. 2) que dans le cas des ferrites Li-Fe-Co la décroissance de l'anisotropie induite due à l'ordonnance à longue distance est beaucoup moindre qu'aux ferrites Li-Fe.

Nous recherches ont regardé aussi la manière dans laquelle varie la décroissance de l'anisotropie induite au TTM aux ferrites de lithium avec l'addition de cobalt dans des concentrations différentes. Puisque les échantillons de ce système ont des teneurs de Fe^{2+} différentes et K_u^{ex} varie forte avec l'addition de cobalt [4] et avec la teneur des ions Fe^{+2} [11], nous

caractériserons la décroissance de l'anisotropie induite par le rapport $\Delta K_u/K_u^{\text{ex}}$.

Dans la fig. 3 on donne la courbe de variation du rapport $\Delta K_u/K_u^{\text{ex}}$ en fonction de la concentration du cobalt y aux ferrites Li-Fe-Co. Le rapport présente une décroissance rapide pour des petites concentrations de cobalt suivi d'une décroissance lente pour des plus grandes concentrations. Ce type de variation peut être expliqué de la manière suivante : aux petites concentrations de cobalt, la composante de l'anisotropie induite K_u^{Li} décroît fortement à cause de l'ordonnance à longue distance des ions de lithium, tandis qu'aux concentrations plus grandes, la composante K_u^{Co} devient prépondérante, étant moins affectée par l'ordonnance à longue distance.

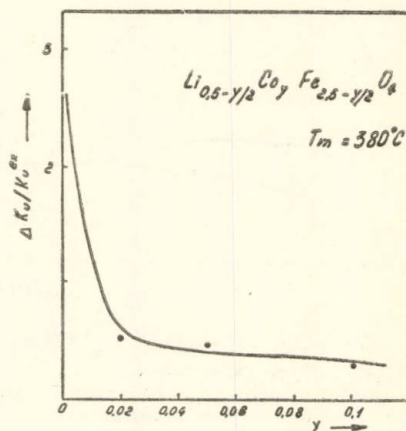


Fig. 3. — Le rapport $\Delta K_u/K_u^{\text{ex}}$ en fonction de la concentration des ions de cobalt.

Pour les ferrites Li-Fe on explique [2] la décroissance de l'anisotropie induite avec la durée de TTM par la décroissance du nombre des configurations anisotropes qui s'ordonnent directionnellement, à cause du processus d'ordonnance à longue distance.

Aux ferrites Li-Fe-Co la décroissance de l'anisotropie induite ne peut pas être expliqué seulement par l'ordonnance à longue distance des ions de lithium. L'ordonnance à longue distance détermine aussi une décroissance de l'anisotropie induite, causé par l'ordre directionnel des ions de cobalt (K_u^{Co}).

L'anisotropie induite au TTM dans un ferromagnétique polycristallin, a d'après Néel [10] l'expression suivante :

$$K_u = \frac{2N\omega(T_i)\omega(T_m)}{15KT_i}$$

ou N reprezintă le nombre de configurations élémentaires qui s'ordonnent directionnellement, $\omega(T_i)$ et $\omega(T_m)$ — l'énergie élémentaire d'anisotropie à la température de traitement, respectivement de mesure, K — la constante de Boltzmann. L'anisotropie induite dépend donc de deux facteurs : du nombre des configurations qui s'ordonnent directionnellement N et de l'énergie élémentaire d'anisotropie ω .

Puisque l'anisotropie induite aux ferrites avec cobalt est causé par l'ordre directionnel des ions uniques de cobalt sur les sites octaédriques [9], même si les ions de cobalt diffusaient, le nombre de configurations anisotropes restaient invariable.

Parce que le nombre de configurations restent invariable, nous supposons que la décroissance de l'anisotropie induite causée par l'ordre des ions de cobalt est due à la décroissance de l'anisotropie élémentaire par suite de l'anisotropie des configurations qui s'ordonnent directionnellement.

Dans le cas des ferrites Li-Fe-Co la décroissance de l'anisotropie induite avec la durée de TTM a deux causes : a) la décroissance du nombre des configurations élémentaires anisotropes, conséquence de l'ordonnance des ions de lithium dans le sous-réseau octaédrique ; b) la décroissance de l'énergie élémentaire d'anisotropie des configurations auxquelles participe le cobalt.

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ANIZOTROPIA INDUSĂ ÎN FERITELE Li-Fe-Co ŞI INFLUENŢA ORDONĂRII IONILOR ASUPRA ACESTEIA

(Rezumat)

Se studiază anizotropia indusă în feritele Li-Fe-Co în domeniul de temperatură 200...400°C. Ordonarea la distanță de tipul 1:3 din subrețeaua octaedrică determină o scădere a anizotropiei induse prin ordonare direcțională. La feritele Li-Fe-Co această scădere a anizotropiei este mai mică decât la feritele Li-Fe. Rezultatele experimentale sînt interpretate pe baza teoriei ordonării direcționale.

ÜBER DEN POLARISIERUNGSVORGANG IN PULSIERENDEN FELDERN VON SEHR NIEDERER FREQUENZ¹⁾

VON

EUGEN NEAGU und EMIL LUCA

1. Einführende Begriffe

Zwecks Information über die Struktur makromolekularen Materialien und ihrer Polarisierung in pulsierenden und Kontinuitätsfeldern ist das Problem der Zeitbestimmung zur Festsetzung der Entspannungspolarisierung, deren Wert Minuten ja sogar Stunden [1], [2] erreichen kann, von ausserordentlichem Interesse. In diesem Sinne würden verschiedene experimentelle Untersuchungen in die Wege geleitet, die das Studium des Polarisierungsvorgangs bei einigen Versuchen mit Plexiglas in pulsierenden Feldern von niederer Frequenz betreffen- und zwar in der Frequenzskala 5 Hz ... 50 kHz [3], [4], die interessante, den Wert und die Dauer der Oberflächenbelastung betreffende Ergebnisse aufwiesen. Die Verwendung noch niedriger Frequenzen, die im Orientierungsvorgang auch polare Gruppierungen von grösseren Ausmassen beeinflussen können, deren Desorientierung sich langsamer vollzieht, erfordert die Ausführung einiger experimentellen Untersuchungen, die Polarisierungsvorgänge im Bereich sehr niederer Frequenzen betreffen.

2. Experimentelle Vorrichtung

Zur Erlangung einiger Impulse von sehr niederer Frequenz, in der Skala 0,005 ... 1 Hz wurde ein verstellbarer unstabiler, transistorisierter Stromkreis verwendet, dessen Signale einen Transistor betrieben, der in

¹⁾ Mitgeteilt an der wissenschaftlichen Tagung anlässlich des Jubiläums der Fakultät für Elektrotechnik, 21—23 Dezember 1972.

der β -Klasse arbeitete und als Belastung einen Relais hatte. Dieses hat die Rolle eines gesteuerten Unterbrechers inne, der die Kontinuitätsspannung, von einer Hochspannungsquelle gespeist, umwandelt in Pulsationsspannung von niedriger Frequenz.

Die zeitliche Schwankung des Stromkreises wird durch einen veränderlichen Widerstand geregelt, indem ein Schalter von 2×12 Positionen verwendet wird, der die Erlangung von 12 feststehenden Wiederholungsfrequenzen erlaubt, bzw. 12 Impulse, deren Dauer genau festgelegt wird und konstant ist. Die Zeit, in der sich die erhaltenen Impulse wiederholen, konnte so von einem Impuls in 300 Sekunden bis zu 2 Impulsen pro Sekunde verändert werden, eine Grenze, die von der Trägheit der mobilen Armatur des Relais auferlegt wird.

Werden zwei Potentiometer mit dem Unterbrechergleitkontakt in Serie geschaltet und normalerweise auf 0-Position eingestellt sind, kann man durch die Veränderung ihrer Werte auch andere unterschiedliche Kombinationen von Wiederholungsperioden ausser den 12 feststehenden erhalten. Die Zeit, in der sich die so erhaltenen Signale wiederholen, ist sehr stabil, da während der Funktionsdauer von 9 Stunden nur Veränderungen bis zu 0,1 Sekunden zu bemerken sind.

Die zwischen 0 und 7 kV einstellbare, ununterbrochene Spannung, wurde von einem stabilisierten Gleichrichter erhalten.

3. Experimentelle Ergebnisse

Die experimentellen Messungen wurden bei Versuchen mit Plexiglas unternommen — in der Form von Platten, die einen Durchmesser von 30 mm und eine Dicke von 4,5 mm halten. Die Platten wurden in einem Wärmeschrank bis auf 155°C erwärmt, dann wurden sie 3 Stunden lang den Einwirkungen des elektrischen Pulsationsfeldes mit sehr niedriger Frequenz ausgesetzt, die bis zur Abkühlung der Platten auf rechterhalten wurde. Zwecks Vergleichsmöglichkeiten wurde parallel zu diesem Vorgang eine zweite Reihe Platten unter denselben Bedingungen der Einwirkung eines kontinuierlichen elektrischen Feldes von derselben Intensität ausgesetzt. Der Polarisierungsgrad und die zeitliche Stabilität wurden durch den Wert der Oberflächenbelastung der Platten mit Hilfe der elektrostatischen Induktionsmethode gemessen und zwar im Augenblick der Entfernung der Platten aus dem Bereich des polarisierenden elektrischen Wirkungsfeldes und nachher in bestimmten Zeitintervallen.

In den Abb. 1, 2, 3 und 4 sind die zeitlichen Veränderungen der elektrischen Belastungen, die auf den Oberflächen der Platten unter der Einwirkung eines elektrischen Kontinuitätsfeldes polarisiert werden, oder unter Einwirkung eines pulsierenden elektrischen Feldes mit Frequenzen von 0,033 Hz (Abb. 3) und 0,016 Hz.

Bei der Analyse der erhaltenen Kurven geht hervor: Für die Frequenz des pulsierenden Feldes von 0,43 Hz ist der ursprüngliche Belas-

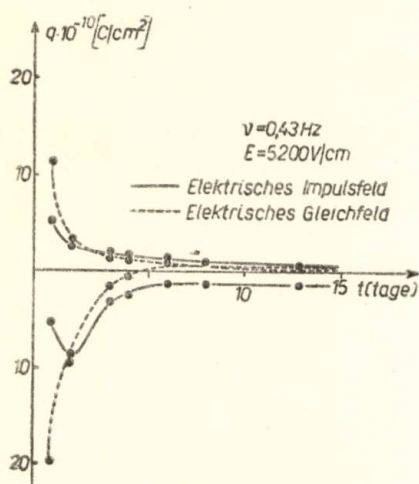


Abb. 1

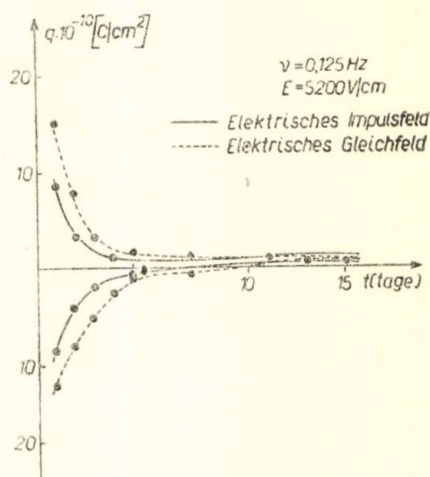


Abb. 2

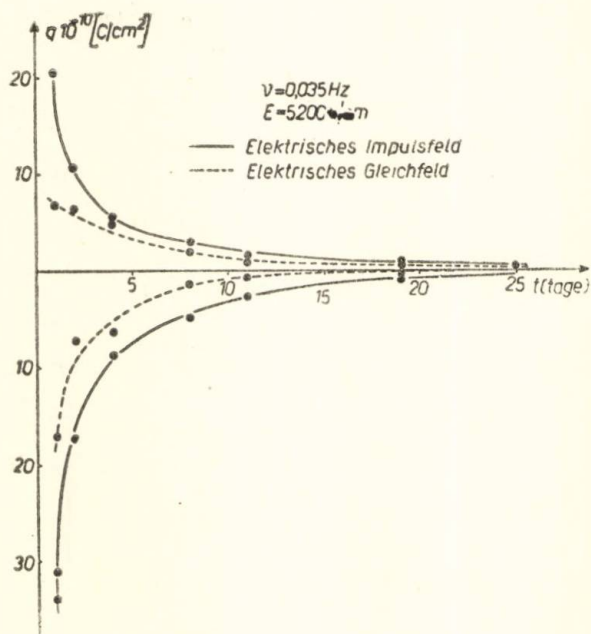


Abb. 8

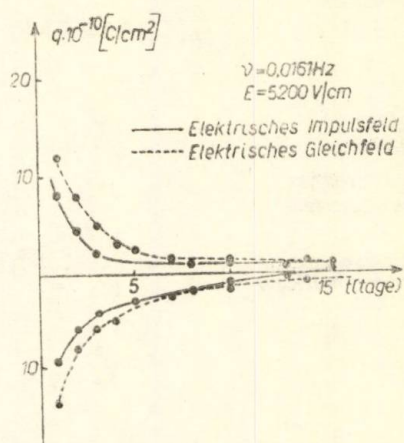


Abb. 4

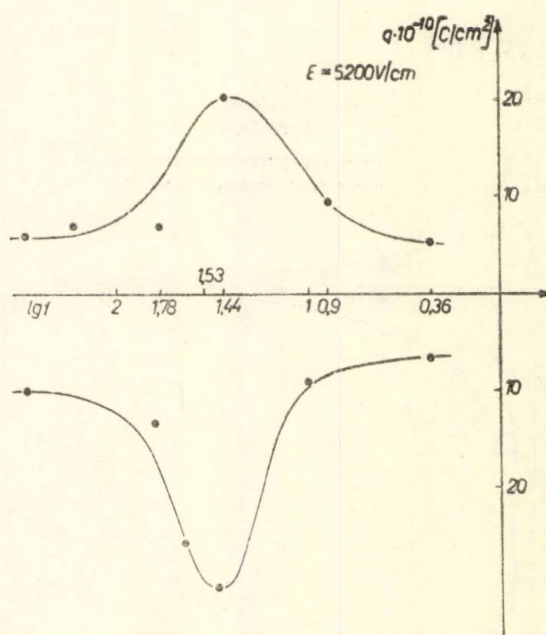


Abb. 5

tungswert auf den Oberflächen der Platten, die in pulsierenden Feldern polarisierten, kleiner als der, welcher der Polarisierung im kontinuierlichen Feld entspricht. Die zeitliche Veränderung der Oberflächenbelastung geht langsamer vonstatten bei den Platten, die im pulsierenden Feld polarisiert wurden, als bei jenen die im kontinuierlichen Feld polarisierten. Dieses beweist, dass während dem ersten Versuch die ständige inhaltliche Polarisierung stärker ist und eine grössere stabilität gewährt. Für die Frequenz des pulsierenden Feldes von 0,125 Hz ist der wert der Oberflächenbelastung auf die im pulsierenden Feld polarisierten Platten auch kleiner als der, welcher den im kontinuierlichen Feld polarisierten Platten entspricht, aber diesmal geht die zeitliche Verminderung der Oberflächenbelastung schneller vonstatten beim ersten Versuch als bei den im kontinuierlichen Feld polarisierten Platten.

Bei den in pulsierenden elektrischen Feldern polarisierten Platten mit einer Frequenz von 0,035 Hz stellt man fest, dass der Wert der ursprünglichen Oberflächenbelastung grösser ist und die zeitliche Verminderung sich langsamer vollzieht, als bei den Platten, die im kontinuierlichen elektrischen Feld polarisiert wurden.

Verkleinert man die Frequenz des pulsierenden elektrischen Feldes noch mehr, bemerkt man eine neuerliche Verminderung des Wertes der ursprünglichen Oberflächenbelastung bei der Polarisierung in pulsierenden Feldern. Bei der Frequenz von 0,0116 Hz dieses Feldes zeigt Abb. 4, dass die Oberflächenbelastung bei den Platten, die im pulsierenden Feld polarisiert wurden, kleiner ist als bei denen, die im kontinuierlichen Feld polarisierten.

Wir glauben, dass die Ergebnisse von der Tatsache bestimmt werden, dass in dem Material, aus dem diese Platten angefertigt wurden, verschiedene polare Gruppen unterschiedlicher Dimensionen existieren, die aufgrund eines Resonanzvorgangs sich nur für eine gewisse Frequenz des pulsierenden elektrischen Feldes orientieren.

Diese Aspekt wird in Abb. 5 hervorgehoben, wo die Veränderung der Oberflächenbelastung, die der Funktion der Frequenz des pulsierenden Feldes vorangeht, dargestellt wird. Ein Höchstmass für die Frequenz von 0,085 Hz, die dem Resonanzvorgang entspricht, ist ersichtlich.

Die Existenz einer Resonanz der Polarisierung, die von der Art und den Dimensionen der polaren Gruppen bestimmt ist, erlaubt im Fall der Verwendung von pulsierenden Feldern von entsprechender Frequenz, ständig polarisierende Platten mit grosser Stabilität zu erhalten.

Die Behandlung dieser theoretischen Aspekte verlangt aber vorher die Durchführung einiger experimentellen Messungen, die an den Wert des Induktionsstroms gebunden sind und in Zukunft durchgeführt werden sollen.

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ASUPRA PROCESULUI DE POLARIZARE ÎN CÎMPURI PULSATORII
DE FOARTE JOASĂ FRECVENȚĂ

(Rezumat)

Se expune metodologia de lucru privind obținerea de impulsuri unidirecționale de foarte joasă frecvență (0,005...2 Hz) și înaltă tensiune. Se studiază procesul de polarizare și posibilitatea de a obține electreți în asemenea cîmpuri pulsatorii. Se compară rezultatele obținute privind valoarea inițială a sarcinei și descreșterea ei în timp, cu rezultatele obținute pentru electreții formați printr-un tratament termic identic în cîmp electric constant.

